

# Mathematical Morphology

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## A few references

- J. Serra, *Image Analysis and Mathematical Morphology*, Academic Press, New-York, 1982.
- J. Serra (Ed.), *Image Analysis and Mathematical Morphology, Part II: Theoretical Advances*, Academic Press, London, 1988.
- P. Soille, *Morphological Image Analysis*, Springer-Verlag, Berlin, 1999.
- L. Najman, H. Talbot (Eds.), *Mathematical Morphology: From Theory to Applications*, ISTE-Wiley, 2010.

# Shape or spatial relations?



# Simplifying and selecting relevant information...

## LES POIRES,

Voici à la vente d'aujourd'hui les Poires que le Commerce de la Capitale...

### Vendues pour payer les 6,000 fr. d'amende du journal le Charivari

Ces fruits sont de la plus grande qualité, et ont été choisis par le Commerce de la Capitale, pour servir de garantie à la vente des Poires de la Capitale. Ils sont vendus à un prix de 1 franc 50 centimes.

Il faut remarquer que les Poires de la Capitale, sont d'une qualité supérieure, et ont été choisis par le Commerce de la Capitale, pour servir de garantie à la vente des Poires de la Capitale.

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Il faut, en outre, remarquer que les Poires de la Capitale, sont d'une qualité supérieure, et ont été choisis par le Commerce de la Capitale, pour servir de garantie à la vente des Poires de la Capitale.

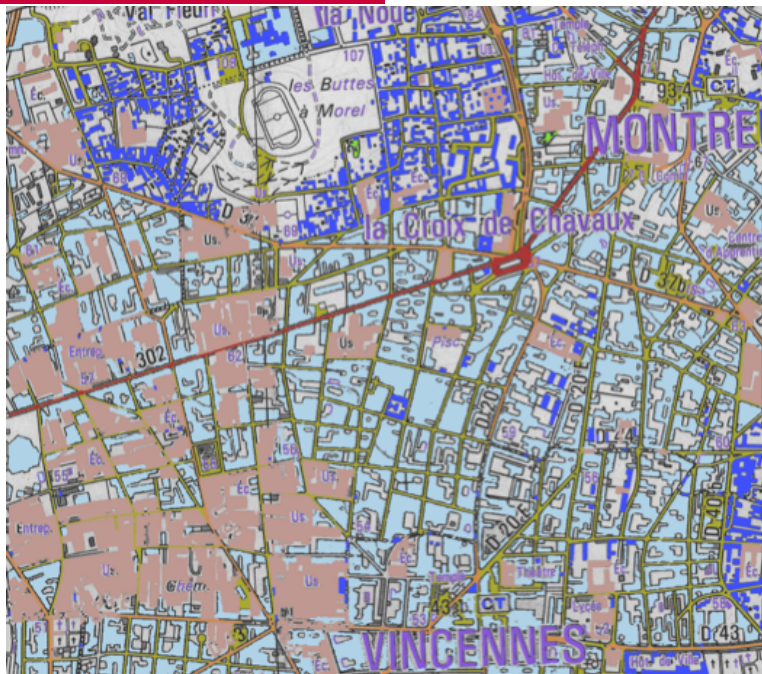
- Origin: study of porous media
- Principle: study of objects (images) based on:
  - shape, geometry, topology
  - grey levels, colors
  - neighborhood information
- Mathematical bases:
  - set theory
  - topology
  - geometry
  - algebra (lattice theory)
  - probabilities, random closed sets
  - functions
- Main characteristics:
  - non linear
  - non invertible
  - strong properties
  - associated algorithms

- filtering
- segmentation
- measures (distances, granulometry, integral geometry, topology, stochastic processes...)
- texture analysis
- shape recognition
- scene interpretation
- ...

Applications in numerous domains

## Example





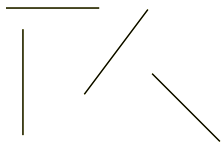
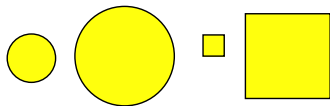
# Four fundamental principles

- 1 Compatibility with translations
- 2 Compatibility with scaling
- 3 Local knowledge
- 4 Continuity (semi-continuity)

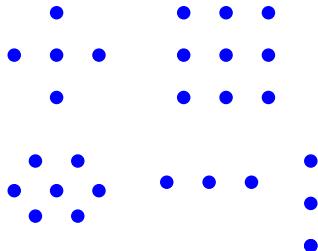
# Structuring element

- shape
- size
- origin (not necessarily in  $B$ )
- examples:

Continuous:



Digital:



# Binary dilation

Minkowski addition:

$$X \oplus Y = \{x + y \mid x \in X, y \in Y\}$$

Binary dilation:

$$\begin{aligned} D(X, B) &= X \oplus B = \{x + y \mid x \in X, y \in B\} \text{ (or } = X \oplus \check{B} \text{ historically)} \\ &= \bigcup_{x \in X} B_x = \{x \in \mathbb{R}^n \mid \check{B}_x \cap X \neq \emptyset\} \end{aligned}$$

( $\check{B}$  = symmetrical of  $B$  with respect to space origin,  $B_x = x + B$ )

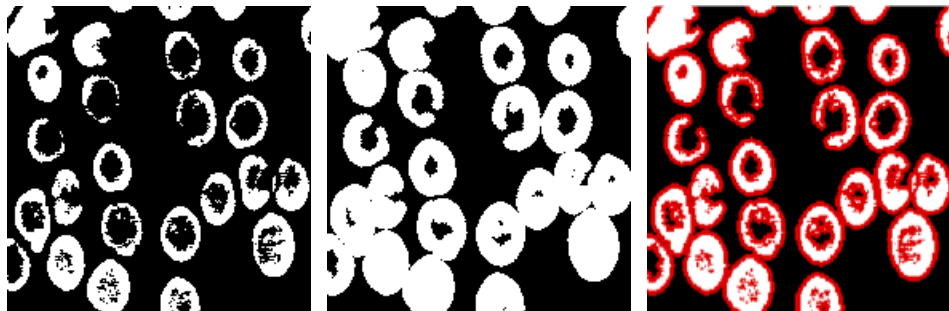
Properties of dilation:

- extensive ( $X \subseteq D(X, B)$ ) iff  $O \in B$ ;
- increasing ( $X \subseteq Y \Rightarrow D(X, B) \subseteq D(Y, B)$ );
- $B \subseteq B' \Rightarrow D(X, B) \subseteq D(X, B')$ ;
- commutes with union, not with intersection:

$$D(X \cup Y, B) = D(X, B) \cup D(Y, B), \quad D(X \cap Y, B) \subseteq D(X, B) \cap D(Y, B);$$

- iterativity property:  $D[D(X, B), B'] = D(X, B \oplus B')$ .

## Example of dilation



$$\begin{aligned} E(X, B) &= \{x \in \mathbb{R}^n \mid B_x \subseteq X\} \\ &= \{x \in \mathbb{R}^n \mid \forall y \in B, x + y \in X\} = X \ominus \check{B}. \end{aligned}$$

## Properties of erosion:

- duality of erosion and dilation with respect to complementation:

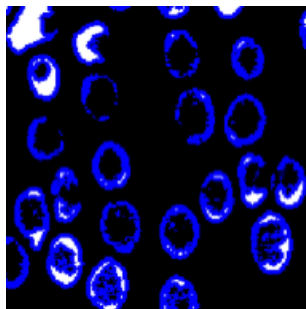
$$E(X, B) = [D(X^C, B)]^C$$

- anti-extensive ( $E(X, B) \subseteq X$ ) iff  $O \in B$ ;
- increasing ( $X \subseteq Y \Rightarrow E(X, B) \subseteq E(Y, B)$ );
- $B \subseteq B' \Rightarrow E(X, B') \subseteq E(X, B)$ ;
- commutes with intersection, not with union:

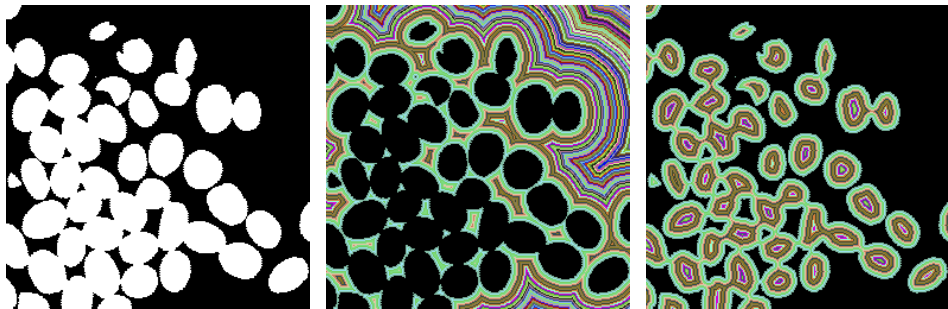
$$E[(X \cap Y), B] = E(X, B) \cap E(Y, B), \quad E[(X \cup Y), B] \supseteq E(X, B) \cup E(Y, B)$$

- iterativity property:  $E[E(X, B), B'] = E(X, B \oplus B')$ .
- $D[E(X, B), B'] \subseteq E[D(X, B'), B]$ .

## Example of erosion



## Links with distances



$$X_B = D[E(X, B), B]$$

## Properties of opening:

- anti-extensive ( $X \supseteq X_B$ );
- increasing ( $X \subseteq Y \Rightarrow X_B \subseteq Y_B$ );
- idempotent ( $(X_B)_B = X_B$ ).

## $\Rightarrow$ Morphological filter

- $B \subseteq B' \Rightarrow X_{B'} \subseteq X_B$ ;
- $(X_n)_{n'} = (X_{n'})_n = X_{\max(n, n')}$  (with  $X_n$  opening with a structuring element of size  $n$ ).

## Example of opening



$$X^B = E[D(X, B), B]$$

## Properties of closing:

- extensive ( $X \subseteq X^B$ );
- increasing ( $X \subseteq Y \Rightarrow X^B \subseteq Y^B$ );
- idempotent ( $(X^B)^B = X^B$ ).

## $\Rightarrow$ Morphological filter

- $B \subseteq B' \Rightarrow X^B \subseteq X^{B'}$ ;
- $(X^n)^{n'} = (X^{n'})^n = X^{\max(n, n')}$ ;
- $X^B = [(X^C)_B]^C$ .

## Example of closing



- choice of the digital grid (both for the image and the structuring element)
- translations on the grid
- same properties

# From sets to functions

- subgraph of a function on  $\mathbb{R}^n =$  subset of  $\mathbb{R}^{n+1}$
- cuts of a function = sets

$$f_\lambda = \{x | f(x) \geq \lambda\}$$

$$D(f_\lambda, B) = [D(f, B)]_\lambda$$

- functional equivalents of set operations:

$$\cup \rightarrow \text{sup} / \vee$$

$$\cap \rightarrow \text{inf} / \wedge$$

$$\subseteq \rightarrow \leq$$

$$\supseteq \rightarrow \geq$$

# Dilation of a function by a flat structuring element

$$\forall x \in \mathbb{R}^n, D(f, B)(x) = \sup\{f(y) \mid y \in \check{B}_x\}$$

Properties of functional dilation:

- extensivity iff  $0 \in B$ ;
- increasingness;
- $D(f \vee g, B) = D(f, B) \vee D(g, B)$ ;
- $D(f \wedge g, B) \leq D(f, B) \wedge D(g, B)$ ;
- iterativity property.

It holds:

$$D(f_\lambda, B) = [D(f, B)]_\lambda$$

## Example of functional dilation



$$\forall x \in \mathbb{R}^n, E(f, B)(x) = \inf\{f(y) \mid y \in B_x\}$$

## Properties of functional erosion:

- functional dilation and erosion are dual operators;
- anti-extensivity iff  $0 \in B$ ;
- increasingness;
- $E(f \vee g, B) \geq E(f, B) \vee E(g, B)$ ;
- $E(f \wedge g, B) = E(f, B) \wedge E(g, B)$ ;
- iterativity property.

## Example of functional erosion



$$f_B = D[E(f, B), B]$$

Properties of functional opening:

- anti-extensive;
- increasing;
- idempotent.

⇒ morphological filter

## Example of functional opening



$$f^B = E[D(f, B), B]$$

Properties of functional closing:

- extensive;
- increasing;
- idempotent.

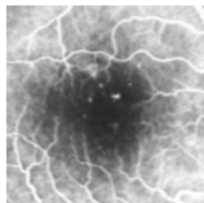
⇒ morphological filter

- duality between opening and closing

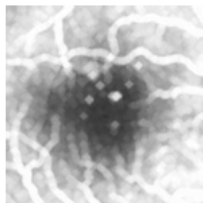
## Example of functional closing



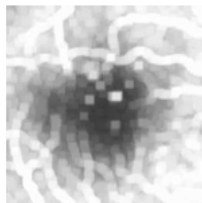
## Example



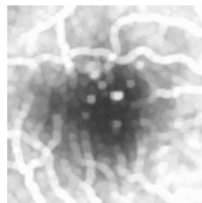
(a)



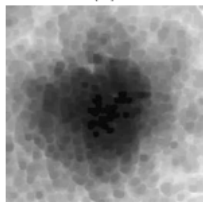
(b)



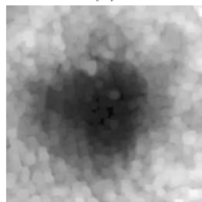
(c)



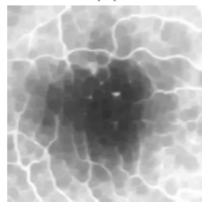
(d)



(e)



(f)



(g)

(a) Original image. Dilation of size 4 using (b) 4-connectivity structuring element, (c) 8-connectivity structuring element, (d) a discrete approximation of a disk. (e) Erosion of size 4 using 4-connectivity structuring element. (f) Opening of size 4. (g) Closing of size 4.

# Structuring functions

Dilation:

$$D(f, g)(x) = \sup_y \{f(y) + g(x - y)\}$$

Erosion:

$$E(f, g)(x) = \inf_y \{f(y) - g(y - x)\}$$

Flat structuring element:

$$g(x) = \begin{cases} 0 & \text{on a compact set } B \\ -\infty & \text{elsewhere} \end{cases}$$

# Some applications of erosion and dilation

## Contrast enhancement

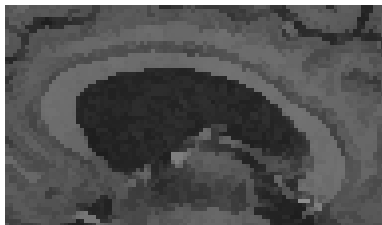
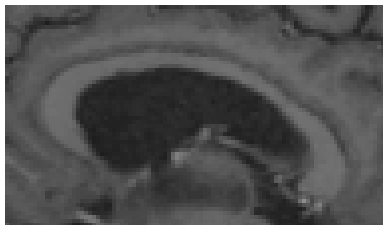


Contrast enhancement: ES 15,  $\alpha = \beta = 0.2$ ,  $\alpha = \beta = 0.3$ ,  $\alpha = \beta = 0.5$



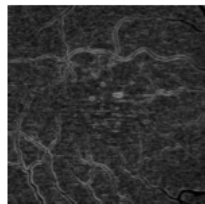
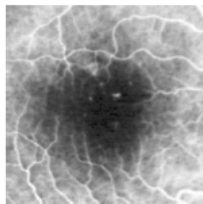
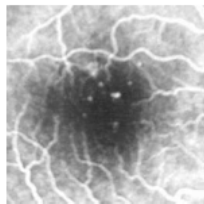
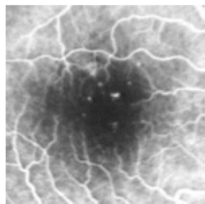
Contrast enhancement: ES 30,  $\alpha = \beta = 0.2$ ,  $\alpha = \beta = 0.3$ ,  $\alpha = \beta = 0.5$





Morphological gradient:  $D_B(x) - E_B(x)$





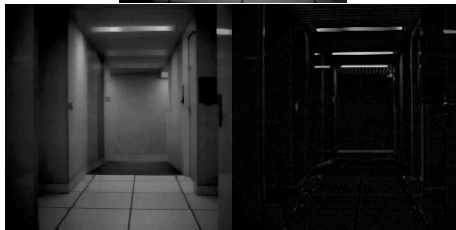
# An application of opening: top-hat transform

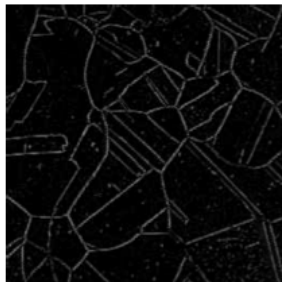
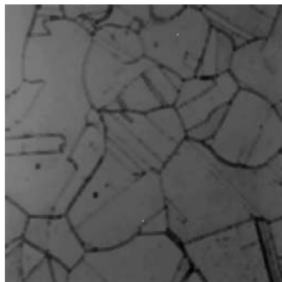
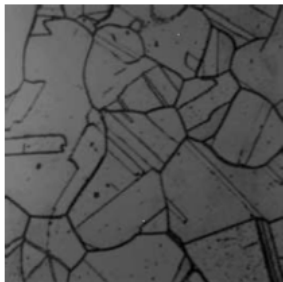
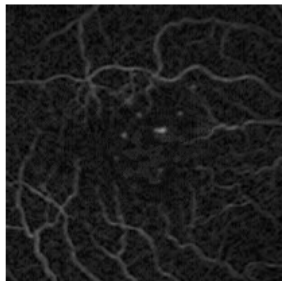
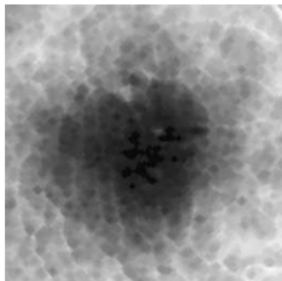
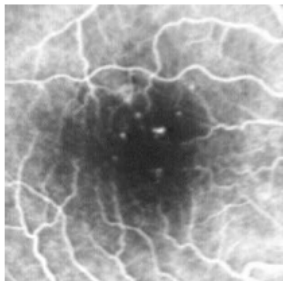
$$f - f_B$$



# An application of opening: top-hat transform

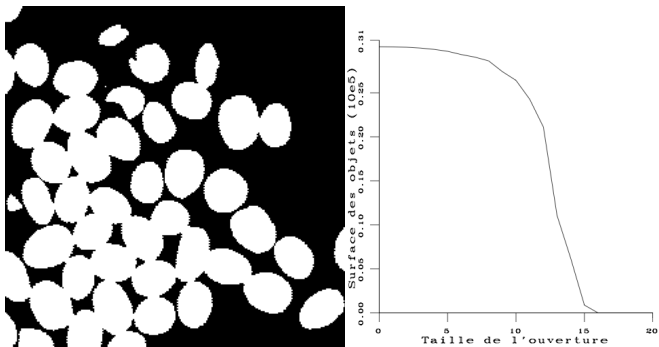
$$f - f_B$$

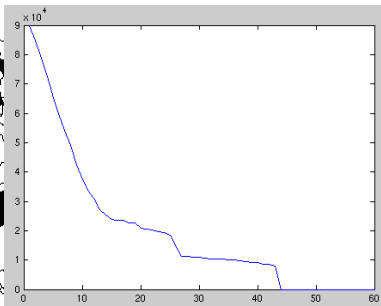
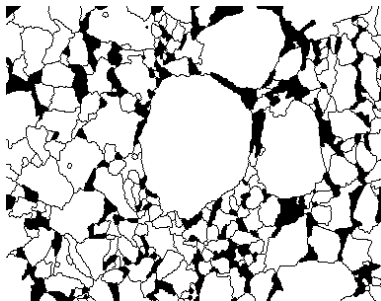




- $\forall X \in \mathcal{A}, \forall \lambda > 0, \phi_\lambda(X) \subseteq X$  ( $\phi_\lambda$  anti-extensive);
- $\forall (X, Y) \in \mathcal{A}^2, \forall \lambda > 0, X \subseteq Y \Rightarrow \phi_\lambda(X) \subseteq \phi_\lambda(Y)$  ( $\phi_\lambda$  increasing);
- $\forall X \in \mathcal{A}, \forall \lambda > 0, \forall \mu > 0 \lambda \geq \mu \Rightarrow \phi_\lambda(X) \subseteq \phi_\mu(X)$  ( $\phi_\lambda$  decreasing with respect to the parameter);
- $\forall \lambda > 0, \forall \mu > 0, \phi_\lambda \circ \phi_\mu = \phi_\mu \circ \phi_\lambda = \phi_{\max(\lambda, \mu)}$ .

$(\phi_\lambda)$  is a granulometry iff  $\phi_\lambda$  is an opening for each  $\lambda$  and the class of subsets  $\mathcal{A}$  which are invariant under  $\phi_\lambda$  is included in the class of subsets which are invariant under  $\phi_\mu$  for  $\lambda \geq \mu$

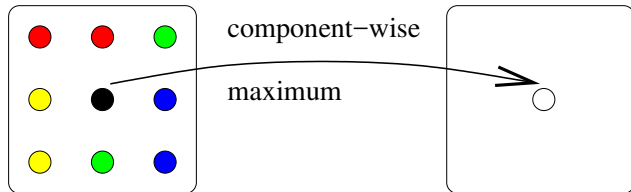




# Vectorial functions (e.g. color images)

- Main difficulty: choice of an ordering
- component-wise max (or min): no good properties

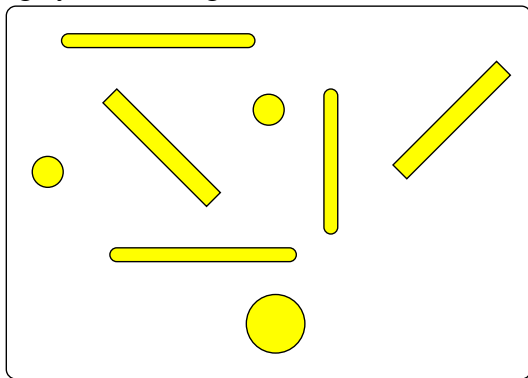
Dilation



# Choice of the structuring element

- depends on what one wants suppress / keep
- shape
- size

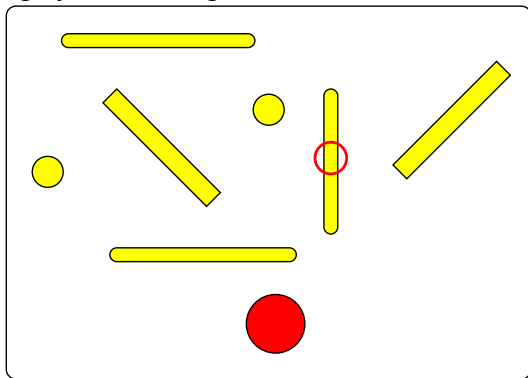
Example: opening by disks or segments?



# Choice of the structuring element

- depends on what one wants suppress / keep
- shape
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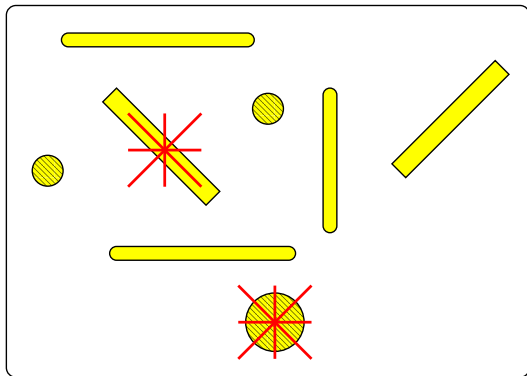
Example: opening by disks or segments?



# Choice of the structuring element

- depends on what one wants suppress / keep
- shape
- size

Example: opening by disks or segments?



Rq: a union of openings is an opening

- Set theory
  - relations ( $\subseteq, \cap, \cup \dots$ )
  - structuring element
- Topology
  - hit-or-miss topology (Fell's topology)
  - myopic topology
  - Hausdorff distance
- Lattice theory
  - adjunctions
  - algebraic operations
- Probability theory
  - $P(A \cap K \neq \emptyset)$
  - random closed sets

# Hit-or-miss topology

- topology on closed subsets
- generated by  $\mathcal{F}^K$  and  $\mathcal{F}_G$  ( $K$  compact and  $G$  open):

$$\mathcal{F}^K = \{F \in \mathcal{F}, F \cap K = \emptyset\}$$

$$\mathcal{F}_G = \{F \in \mathcal{F}, F \cap G \neq \emptyset\}$$

- convergence in  $\mathcal{F}$ :  $(F_n)_{n \in \mathbb{N}}$  converges towards  $F \in \mathcal{F}$  if:

$$\begin{cases} \forall G \in \mathcal{G}, G \cap F \neq \emptyset, \exists N, \forall n \geq N, G \cap F_n \neq \emptyset \\ \forall K \in \mathcal{K}, K \cap F = \emptyset, \exists N', \forall n \geq N', K \cap F_n = \emptyset \end{cases}$$

**Union is continuous** from  $\mathcal{F} \times \mathcal{F}$  in  $\mathcal{F}$  but intersection is not



semi-continuity

# Semi-continuity

$$f : \Omega \rightarrow \mathcal{F}$$

- $f$  upper semi-continuous (u.s.c.) if  $\forall \omega \in \Omega$  and  $\forall (\omega_n)_{n \in \mathbb{N}} \in \Omega$  converging towards  $\omega$  :

$$\overline{\lim} f(\omega_n) \subseteq f(\omega)$$

- $f$  lower semi-continuous (l.s.c.) if:

$$\underline{\lim} f(\omega_n) \supseteq f(\omega)$$

$\overline{\lim}/\underline{\lim} = \cup/\cap$  of adherence points

$f$  continuous iff  $f$  l.s.c. and u.s.c.

Intersection is u.s.c.

# Properties of morphological operations

- the dilation of a closed set by a compact set is continuous
- the dilation of a compact set by a compact set is continuous
- $(F, K) \mapsto E(F, K)$  u.s.c.
- $(K', K) \mapsto E(K', K)$  u.s.c.
- $(F, K) \mapsto F_K$  u.s.c.
- $(K', K) \mapsto K'_K$  u.s.c.
- $(F, K) \mapsto F^K$  u.s.c.
- $(K', K) \mapsto K'^K$  u.s.c.

- generated by:

$$\mathcal{K}_G^F = \{K \in \mathcal{K}, K \cap F = \emptyset, K \cap G \neq \emptyset\}$$

$$(F \in \mathcal{F}, G \in \mathcal{G})$$

- finer than the topology induced on  $\mathcal{K}$  by the hit-or-miss topology
- equivalent on  $\mathcal{K} \setminus \emptyset$  to the topology induced by the Hausdorff distance

$$\delta(K, K') = \max\left\{\sup_{x \in K} d(x, K'), \sup_{x' \in K'} d(x', K)\right\}$$

$$\text{Rq: } \delta(K, K') = \inf\{\varepsilon, K \subseteq D(K', B^\varepsilon), K' \subseteq D(K, B^\varepsilon)\}$$

# Algebraic framework: complete lattices

- Lattice:  $(\mathcal{T}, \leq)$  ( $\leq$  ordering) such that  $\forall(x, y) \in \mathcal{T}, \exists x \vee y$  and  $\exists x \wedge y$
- Complete lattice: every family of elements (finite or not) has a smallest upper bound and a largest lower bound
- $\Rightarrow$  contains a smallest element  $0$  and a largest element  $1$ :

$$0 = \bigwedge \mathcal{T} = \bigvee \emptyset \text{ et } 1 = \bigvee \mathcal{T} = \bigwedge \emptyset$$

- Examples of complete lattices:

- $(\mathcal{P}(E), \subseteq)$ : complete lattice, Boolean (complemented and distributive):

$$\forall x, \exists x^C, x \wedge x^C = 0 \text{ and } x \vee x^C = 1$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

- $(\mathcal{F}(\mathbb{R}^d), \subseteq)$

- functions of  $\mathbb{R}^n$  in  $\overline{\mathbb{R}}$  for the ordering  $\leq$ :

$$f \leq g \Leftrightarrow \forall x \in \mathbb{R}^n, f(x) \leq g(x)$$

- partitions

# Semi-continuity of functions

- u.s.c. :

$$\forall t > f(x), \exists V(x), \forall y \in V(x), t > f(y)$$

( $V(x)$  neighborhood of  $x$  in  $\mathbb{R}^n$ )

- l.s.c. :

$$\forall t < f(x), \exists V(x), \forall y \in V(x), t < f(y)$$

- a function is u.s.c. iff its sub-graph is closed
- topology on the space of u.s.c. functions = topology induced by the hit-or-miss topology on  $\mathcal{F}(\mathbb{R}^n \times \overline{\mathbb{R}})$
- the set of u.s.c. functions of  $\mathbb{R}^n$  in  $\overline{\mathbb{R}}$  is a complete lattice for  $\leq$ :

$$f \leq g \Leftrightarrow SG(f) \subseteq SG(g)$$

# Algebraic dilation and erosion

Complete lattice  $(\mathcal{T}, \leq)$

Algebraic dilation:

$$\forall (x_i) \in \mathcal{T}, \delta(\bigvee_i x_i) = \bigvee_i \delta(x_i)$$

Algebraic erosion:

$$\forall (x_i) \in \mathcal{T}, \varepsilon(\bigwedge_i x_i) = \bigwedge_i \varepsilon(x_i)$$

Properties:

- $\delta(0) = 0$  (in  $\mathcal{P}(E)$ ,  $0 = \emptyset$ )
- $\varepsilon(I) = I$  (in  $\mathcal{P}(E)$ ,  $I = E$ )
- $\delta$  increasing
- $\varepsilon$  increasing
- in  $\mathcal{P}(\mathbb{R}^n)$ ,  $\delta(X) = \bigcup_{x \in X} \delta(\{x\})$

# Adjunctions

$(\varepsilon, \delta)$  adjunction on  $(\mathcal{T}, \leq)$ :

$$\forall(x, y), \delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y)$$

Properties:

- $\delta(0) = 0$  and  $\varepsilon(I) = I$
- $(\varepsilon, \delta)$  adjunction  $\Rightarrow \varepsilon =$  algebraic erosion and  $\delta =$  algebraic dilation
- $\delta$  increasing = algebraic dilation iff  $\exists \varepsilon$  such that  $(\varepsilon, \delta)$  is an adjunction  
 $\Rightarrow \varepsilon =$  algebraic erosion and  $\varepsilon(x) = \bigvee\{y \in \mathcal{T}, \delta(y) \leq x\}$
- $\varepsilon$  increasing = algebraic erosion iff  $\exists \delta$  such that  $(\varepsilon, \delta)$  is an adjunction  
 $\Rightarrow \delta =$  algebraic dilation and  $\delta(x) = \bigwedge\{y \in \mathcal{T}, \varepsilon(y) \geq x\}$
- $\varepsilon\delta \geq Id$
- $\delta\varepsilon \leq Id$
- $\varepsilon\delta\varepsilon = \varepsilon$
- $\delta\varepsilon\delta = \delta$
- $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$  and  $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$

- On the lattice of the subsets of  $\mathbb{R}^n$  or  $\mathbb{Z}^n$ , with inclusion:

$$\delta(X) = \cup_{x \in X} \delta(\{x\})$$

- + invariance under translation  $\Rightarrow \exists B, \delta(X) = D(X, B)$
- Same result on the lattice of functions.
- Similar results for erosion.

# Algebraic opening and closing

- **Algebraic opening:**  $\gamma$  increasing, idempotent and anti-extensive
- **Algebraic closing:**  $\varphi$  increasing, idempotent and extensive
- **Examples:**  $\gamma = \delta\varepsilon$  and  $\varphi = \varepsilon\delta$  with  $(\varepsilon, \delta) = \text{adjunction}$
- **Invariance domain:**  $\text{Inv}(\varphi) = \{x \in \mathcal{T}, \varphi(x) = x\}$
- $\gamma$  opening  $\Rightarrow \gamma(x) = \bigvee\{y \in \text{Inv}(\gamma), y \leq x\}$
- $\varphi$  closing  $\Rightarrow \varphi(x) = \bigwedge\{y \in \text{Inv}(\varphi), x \leq y\}$
- $(\gamma_i)$  openings  $\Rightarrow \bigvee_i \gamma_i$  opening
- $(\varphi_i)$  closings  $\Rightarrow \bigwedge_i \varphi_i$  closing
- $\gamma_1$  and  $\gamma_2$  openings  $\Rightarrow$  equivalence between:
  - 1  $\gamma_1 \leq \gamma_2$
  - 2  $\gamma_1\gamma_2 = \gamma_2\gamma_1 = \gamma_1$
  - 3  $\text{Inv}(\gamma_1) \subseteq \text{Inv}(\gamma_2)$
- $\varphi_1$  and  $\varphi_2$  closings  $\Rightarrow$  equivalence between:
  - 1  $\varphi_2 \leq \varphi_1$
  - 2  $\varphi_1\varphi_2 = \varphi_2\varphi_1 = \varphi_1$
  - 3  $\text{Inv}(\varphi_1) \subseteq \text{Inv}(\varphi_2)$

Filter = increasing and idempotent operator

## Examples

- openings  $\gamma$  and  $\bigvee_i \gamma_i$  (anti-extensive filters)
- closings  $\varphi$  and  $\bigwedge_i \varphi_i$  (extensive filters)

**Theorem on filter composition**  $\varphi$  and  $\psi$  such that  $\varphi \geq \psi$ :

- $\varphi \geq \varphi\psi\varphi \geq \varphi\psi \vee \psi\varphi \geq \varphi\psi \wedge \psi\varphi \geq \psi\varphi\psi \geq \psi$
- $\varphi\psi$ ,  $\psi\varphi$ ,  $\varphi\psi\varphi$  and  $\psi\varphi\psi$  are filters
- $Inv(\varphi\psi\varphi) = Inv(\varphi\psi)$  and  $Inv(\psi\varphi\psi) = Inv(\psi\varphi)$
- $\varphi\psi\varphi$  is the smallest filter which is largest than  $\varphi\psi \vee \psi\varphi$

## Example: alternate sequential filters

- openings  $\gamma_i$  and closings  $\varphi_i$  such that:

$$i \leq j \Rightarrow \gamma_j \leq \gamma_i \leq Id \leq \varphi_i \leq \varphi_j$$

- Theorem on filter composition  $\Rightarrow m_i = \gamma_i \varphi_i$ ,  $n_i = \varphi_i \gamma_i$ ,  $r_i = \varphi_i \gamma_i \varphi_i$  and  $s_i = \gamma_i \varphi_i \gamma_i$  are filters
- Alternate sequential filters:

$$M_i = m_i m_{i-1} \dots m_2 m_1$$

$$N_i = n_i n_{i-1} \dots n_2 n_1$$

$$R_i = r_i r_{i-1} \dots r_2 r_1$$

$$S_i = s_i s_{i-1} \dots s_2 s_1$$

- Property:  $i \leq j \Rightarrow M_j M_i = M_j$ ,  $N_j N_i = N_j$ , ...

# Morphological alternate sequential filters

$$\left( \dots \left( \left( f_{B_1} \right)^{B_1} \right)_{B_2} \right)^{B_2} \dots_{B_n} \right)^{B_n}$$



# Morphological alternate sequential filters

$$\left( \dots \left( \left( f_{B_1} \right)^{B_1} \right)_{B_2} \right)^{B_2} \dots_{B_n} \right)^{B_n}$$



# Comparison of filters on an image with Gaussian noise



Original image



Gaussian noise (variance 20)



$3 \times 3$  mean



$7 \times 7$  mean



Gaussian filter with variance 0.75

Gaussian filter with variance 4.08



Noisy image



Nagao filter



$3 \times 3$  median



$7 \times 7$  median



Noisy image



Alternate sequential filter 1



Alternate sequential filter 2



Alternate sequential filter 3



Original image



Gaussian noise (variance 120)



$3 \times 3$  mean



$7 \times 7$  mean



Gaussian filter with variance 0.75

Gaussian filter with variance 4.08



Noisy image



Nagao filter



$3 \times 3$  median



$7 \times 7$  median



Noisy image



Alternate sequential filter 1



Alternate sequential filter 2



Alternate sequential filter 3

# Comparison of filters on an image with impulse noise



Original image



Impulse noise (intensity 2%)



$3 \times 3$  mean



$7 \times 7$  mean



Gaussian filter with variance 0.75

Gaussian filter with variance 4.08



Noisy image



Nagao filter



$3 \times 3$  median



$7 \times 7$  median



Noisy image



Alternate sequential filter 1



Alternate sequential filter 2



Alternate sequential filter 3



Original image



Impulse noise (intensity 10%)



$3 \times 3$  mean



Gaussian filter with variance 0.75



$7 \times 7$  mean



Gaussian filter with variance 4.08



Noisy image



Nagao filter



$3 \times 3$  median



$7 \times 7$  median



Noisy image



Alternate sequential filter 1



Alternate sequential filter 2

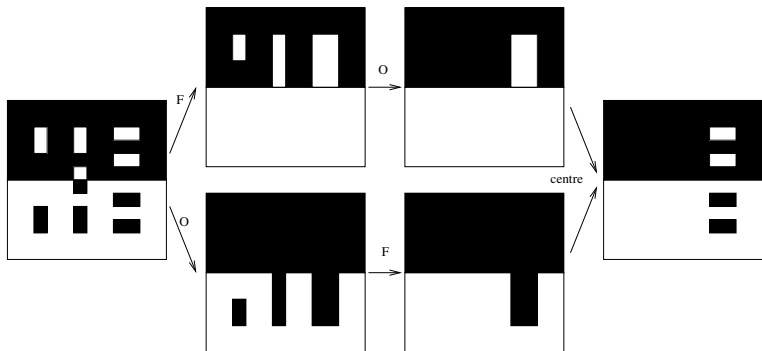


Alternate sequential filter 3

- Operators which are independent of the local contrast, acting similarly on bright and dark areas.
- Example: [morphological center](#)

$$\text{Median}[f(x), \psi_1(f)(x), \psi_2(f)(x)]$$

- More generally, for operators  $\{\psi_1, \psi_2, \dots, \psi_n\}$ :  $(Id \vee \wedge_i \psi_i) \wedge \vee_i \psi_i$
- For instance  $\psi_1(f) = \gamma\varphi(f) = (f^B)_B$ ,  $\psi_2 = \varphi\gamma(f) = (f_B)^B$



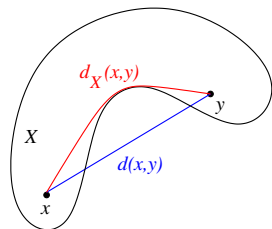
# Geodesic operators

**Geodesic distance**, conditional to  $X$ :  $d_X$

- if  $X$  is closed, there exists a geodesic arc for any pair of points of  $X$
- unique if  $X$  is simply connected
- $X$  convex  $\Leftrightarrow d_X = d$

**Geodesic ball**:  $B_X(x, r) = \{y \in X \mid d_X(x, y) \leq r\}$

Rq:  $B_X(x, r) \subseteq B(x, r)$



**Geodesic dilation**:

$$D_X(Y, B_r) = \{x \in \mathbb{R}^n \mid B_X(x, r) \cap Y \neq \emptyset\} = \{x \in \mathbb{R}^n \mid d_X(x, Y) \leq r\}$$

**Geodesic erosion**:

$$E_X(Y, B_r) = \{x \in \mathbb{R}^n \mid B_X(x, r) \subseteq Y\} = X \setminus D_X(X \setminus Y, B_r)$$

**Geodesic opening and closing**: by composition

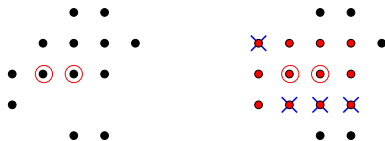
# Properties and reconstruction

## Properties:

- similar as in the Euclidean case
- $D_X(Y, B_r) \subseteq D(Y, B_r)$
- $D_X(Y, B_r) = \bigcap_{n=1}^{\infty} [(Y \oplus \frac{r}{n} B) \cap X]^n$

## Digital case:

$$D_X(Y, B_r) = [D(Y, B_1) \cap X]^r$$

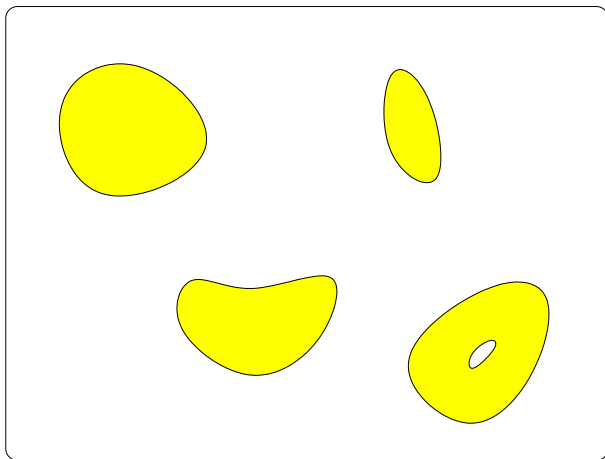


## Reconstruction:

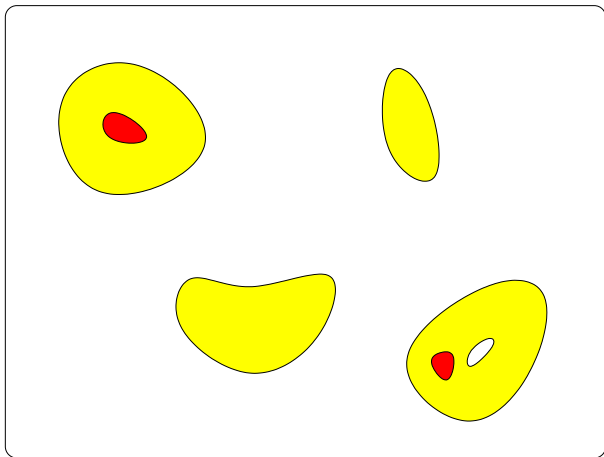
$$[D(Y, B_1) \cap X]^{\infty} = D_X^{\infty}(Y)$$

= connected components of  $X$  which intersect  $Y$

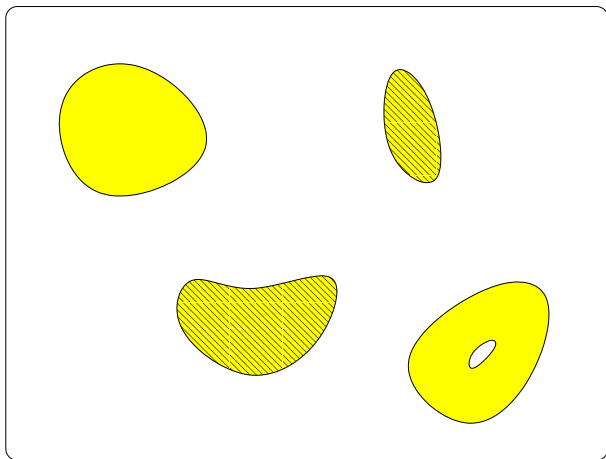
## Binary reconstruction: example



## Binary reconstruction: example



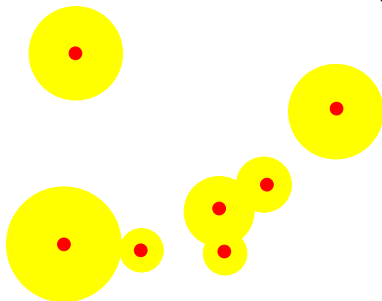
## Binary reconstruction: example



$$EU(X) = \cup_n \{E(X, B_n) \setminus R[E(X, B_{n+1}); E(X, B_n)]\}$$

- $E(X, B_n)$ : erosion of  $X$  by a structuring element of size  $n$
- $R[Y; Z]$ : connected components of  $Z$  having a non-empty intersection with  $Y$

= set of regional maxima of the distance function  $d(x, X^C)$ .



# Geodesic operators on functions

$$X_1 \subseteq X_2 \text{ and } Y_1 \subseteq Y_2 \Rightarrow D_{X_1}(Y_1, B_r) \subseteq D_{X_2}(Y_1, B_r) \subseteq D_{X_2}(Y_2, B_r)$$

$\Rightarrow$  **Extension to functions**, for  $f \leq g$ , cut by cut:

$$[D_g(f, B_r)]_\lambda = D_{g_\lambda}(f_\lambda, B_r)$$

(with  $f_\lambda = \{x, f(x) \geq \lambda\}$ )

**Digital case:**

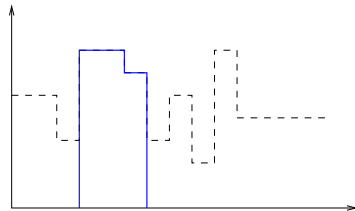
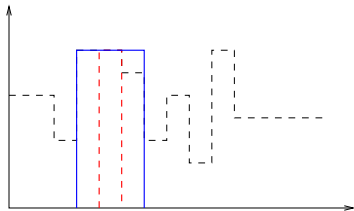
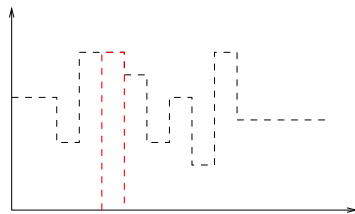
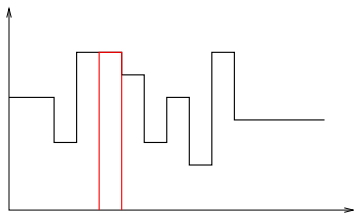
$$D_g(f, B_r) = [D(f, B_1) \wedge g]^r$$

$$E_g(f, B_r) = [E(f, B_1) \vee g]^r$$

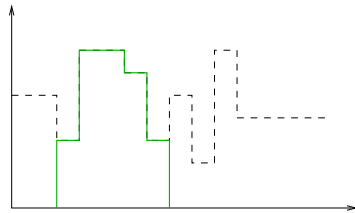
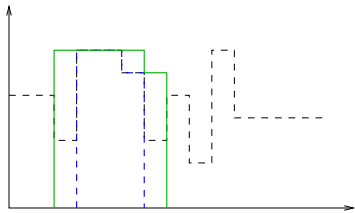
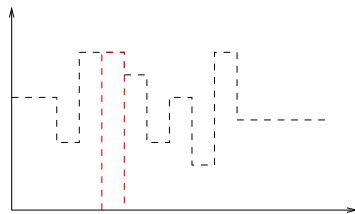
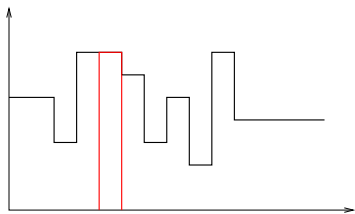
**Numerical reconstruction** of  $f$  (marker function) in  $g$ :

- by dilation  $D_g(f, B_\infty) = D_g^\infty(f)$ : opening
- by erosion  $E_g(f, B_\infty)$ : closing
- opening by reconstruction:  $D_f^\infty(f_B)$  (flat areas whose contours are some contours of the original image  $\Rightarrow$  compression)

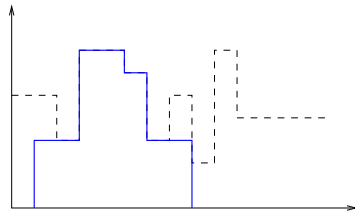
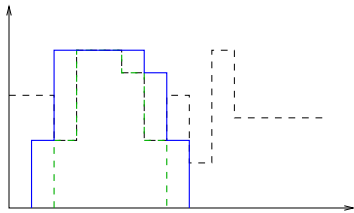
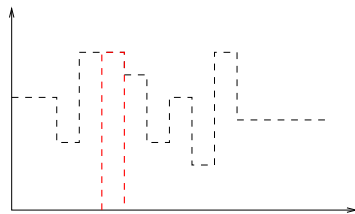
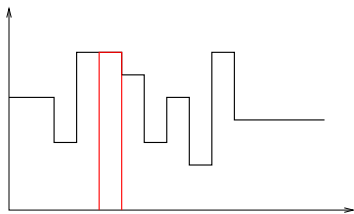
# Numerical reconstruction: example



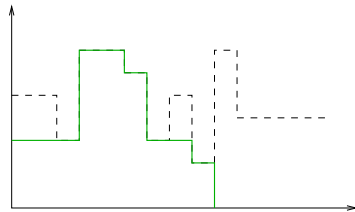
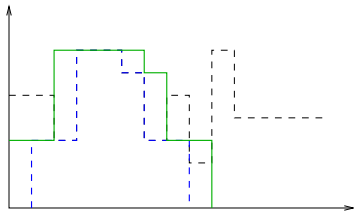
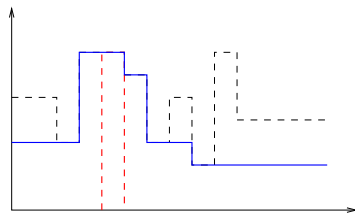
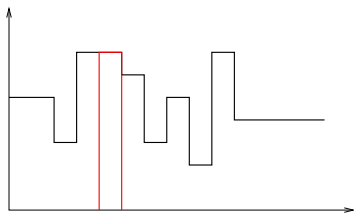
# Numerical reconstruction: example



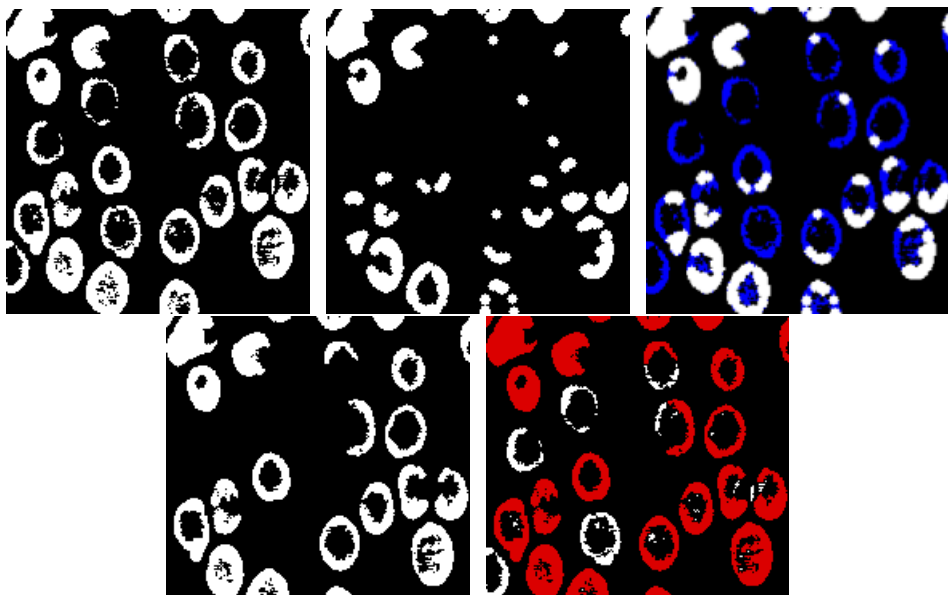
# Numerical reconstruction: example

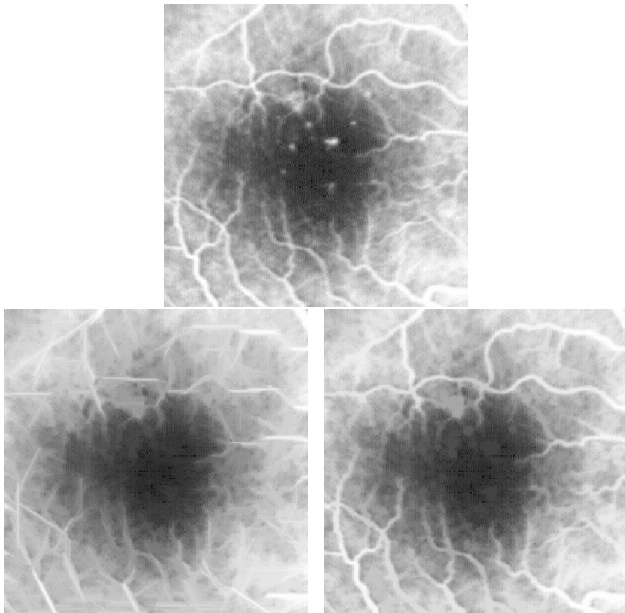


# Numerical reconstruction: example



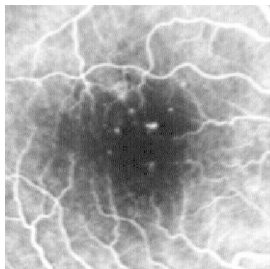
## Opening by reconstruction: examples



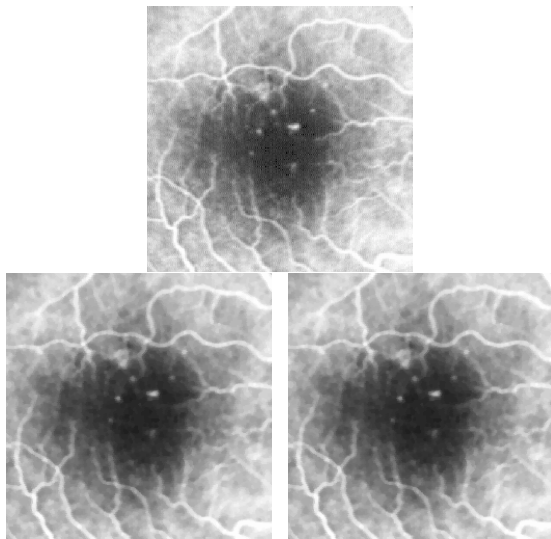


Union of openings by segments of length 20 and reconstruction

# Application to alternate sequential filters

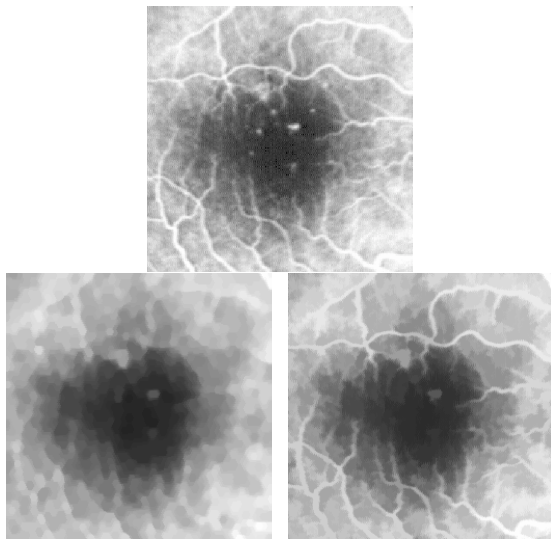


## Application to alternate sequential filters



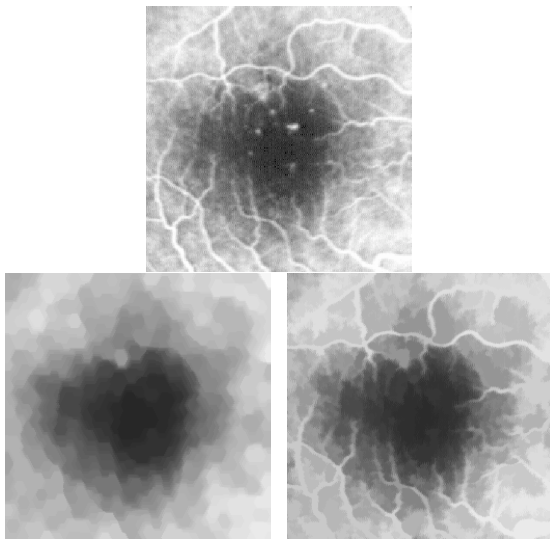
ASF with an hexagon (maximal size = 1) - Right: with reconstruction

## Application to alternate sequential filters



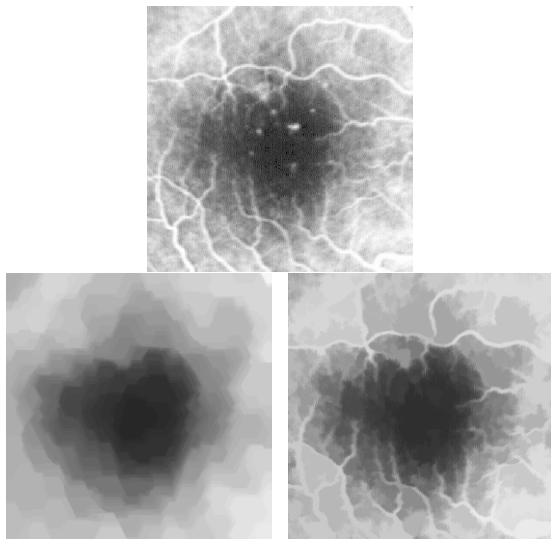
ASF with an hexagon (maximal size = 3)

## Application to alternate sequential filters



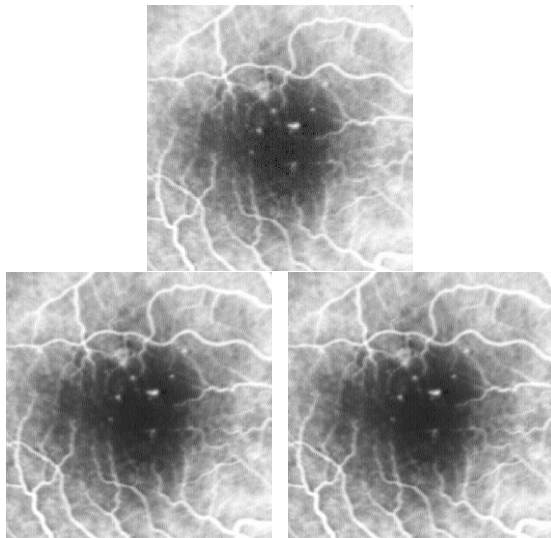
ASF with an hexagon (maximal size = 5)

## Application to alternate sequential filters



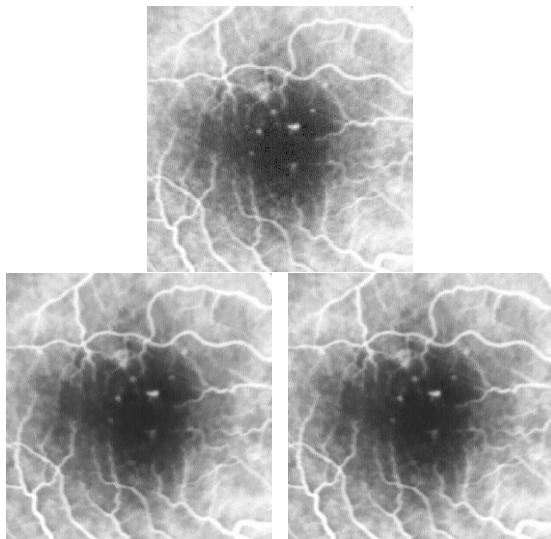
ASF with an hexagon (maximal size = 9)

## Application to alternate sequential filters



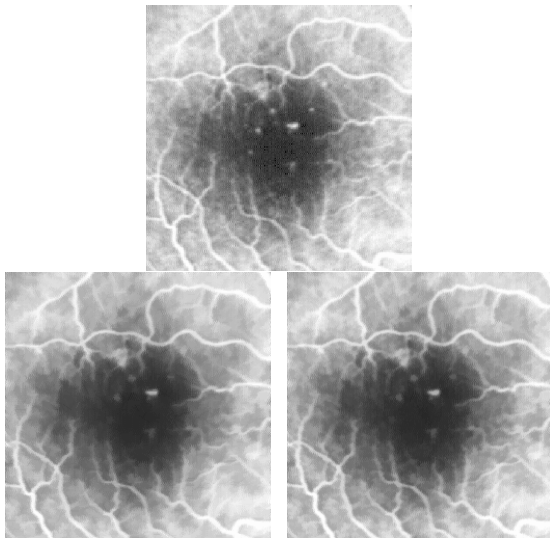
ASF with segments (maximal size = 1)

## Application to alternate sequential filters



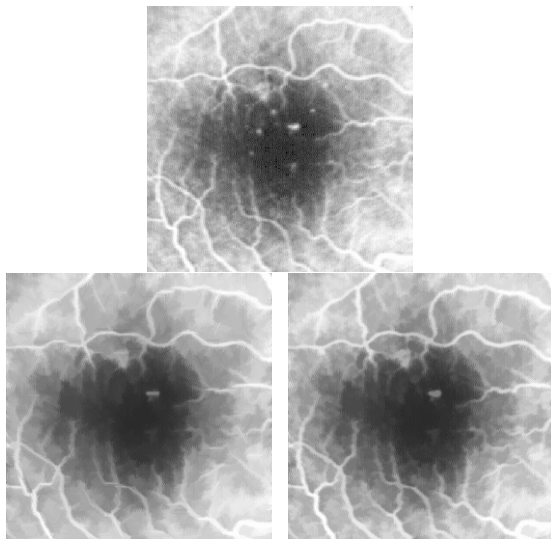
ASF with segments (maximal size = 3)

## Application to alternate sequential filters



ASF with segments (maximal size = 5)

## Application to alternate sequential filters



ASF with segments (maximal size = 9)

$X$  regional maximum of  $f$  if

$$\forall x \in X, f(x) = \lambda \text{ et } X = CC(f_\lambda)$$

Computation of regional maxima:

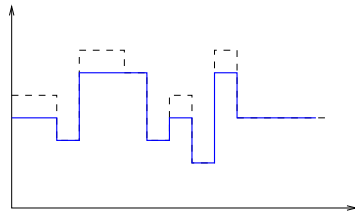
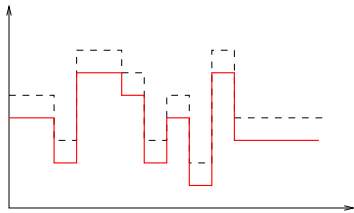
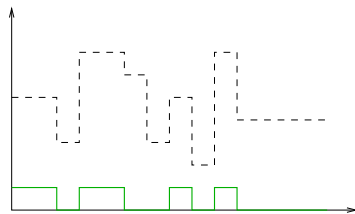
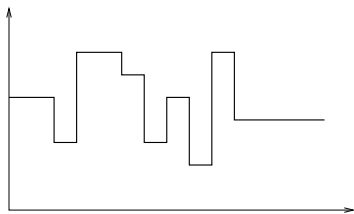
$$f - D_f^\infty(f - 1)$$

$h$ -maxima (gray level dynamics): regional maxima of

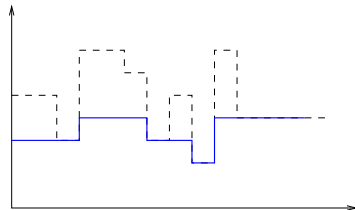
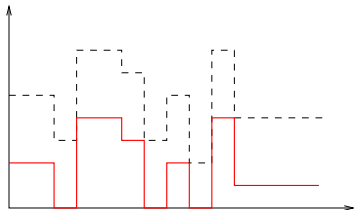
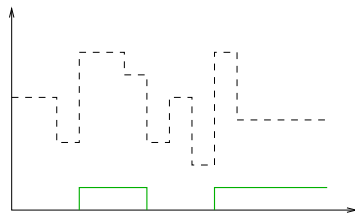
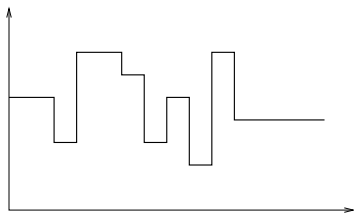
$$D_f^\infty(f - h)$$

$\Rightarrow$  robust maxima

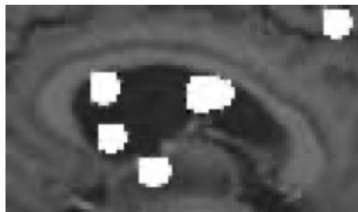
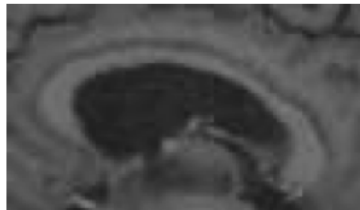
# Regional maxima: example



# Robust maxima: example



## Regional minima: example



## Skeleton by influence zones

$$X = \bigcup_i X_i$$

Influence zone of  $X_i$  in  $X^C$  :

$$ZI(X_i) = \{x \in X^C \mid d(x, X_i) < d(x, X \setminus X_i)\}$$

Skeleton by influence zones:

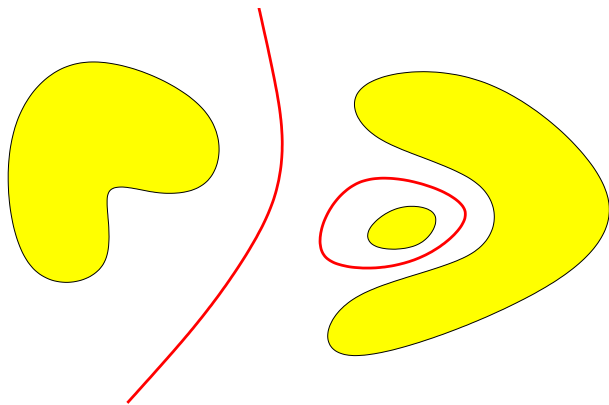
$$\text{Skiz}(X) = \left(\bigcup_i ZI(X_i)\right)^C$$

= generalized Voronoï diagram

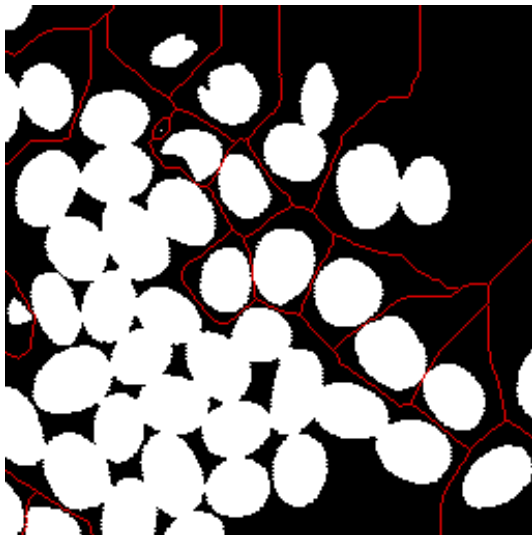
Properties:

- $\text{Skiz}(X) \subseteq \text{Skel}(X^C)$
- Skiz is not necessarily connected (even if  $X^C$  is)

## Skeleton by influence zones: examples



## Skeleton by influence zones: examples



# Geodesic skeleton by influence zones

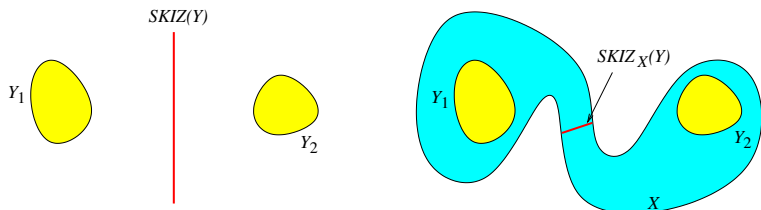
$$Y = \cup_i Y_i$$

Geodesic influence zone of  $Y_i$  conditionally to  $X$  :

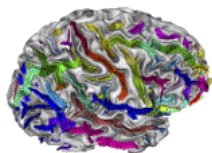
$$ZI_X(Y_i) = \{x \in X, d_X(x, Y_i) < d_X(x, Y \setminus Y_i)\}$$

Geodesic skeleton by influence zones:

$$SKIZ_X(Y) = X \setminus \bigcup_i ZI_X(Y_i)$$

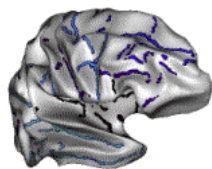


# Cortex segmentation (PhD of Arnaud Cachia)

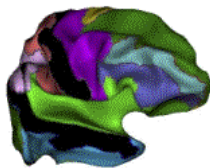


Segmentation et reconnaissance  
automatique des **sillons**

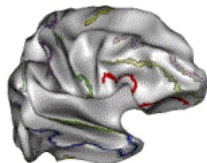
*Rivière00*



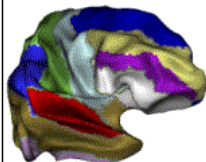
Définition sur la surface corticale des **sillons-frontières**



Calcul des **zones d'influences  
sulcales**

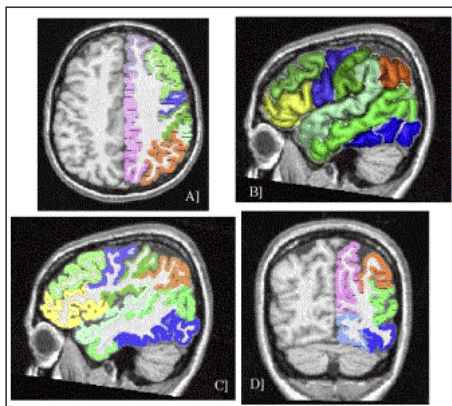
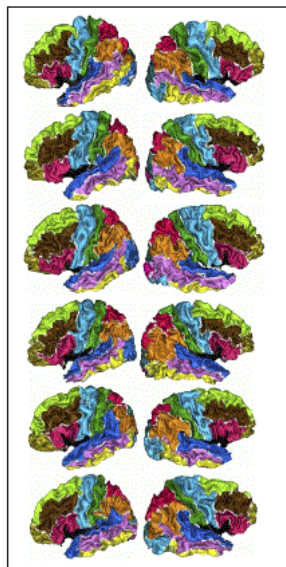


Définition des **graines gyrales**  
(extraction et sélection des  
frontières)



Parcellisation en **gyri**  
(2D et 3D)

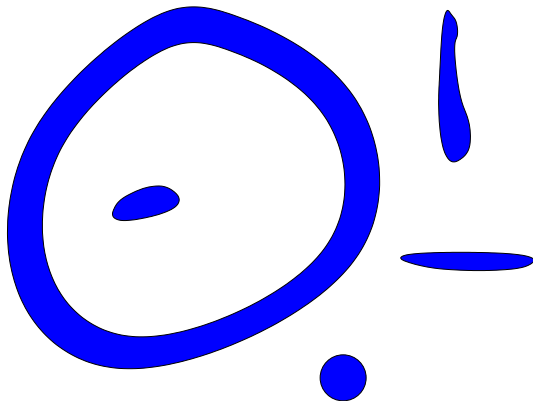
# Cortex segmentation (PhD of Arnaud Cachia)

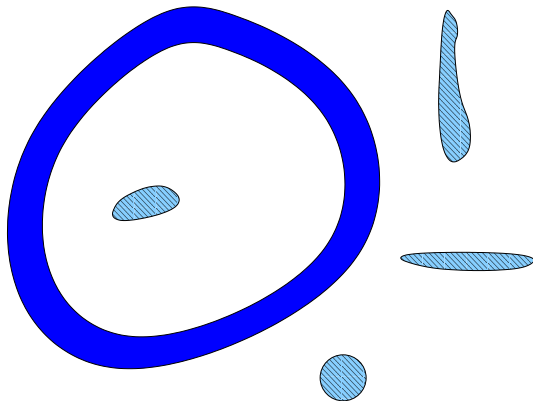


Parcellisation volumique  
(diagramme de Voronoï calculé  
dans le ruban cortical 3D)

- Objective: make the image simpler
- Morphological filter that
  - preserves contours
  - is independent of the contrast
  - acts on connected components
- First example: surfacic opening

$$\gamma_{\lambda}(f) = \bigvee_i \{\gamma_{B_i}(f) \mid B_i \text{ connected and } S(B_i) = \lambda\}$$





# Connected filters on gray level images

- For increasing operations
- Cut by cut computation

$$T_h(f) = \{x \mid f(x) \geq h\}$$

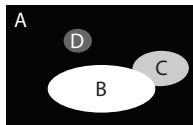
$$(\gamma^A(f))(x) = \sup\{h \mid x \in \Gamma^A(T_h(f))\}$$



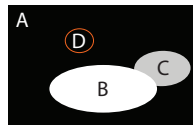
Surfacic opening      Surfacic closing

- More complex for non increasing operators

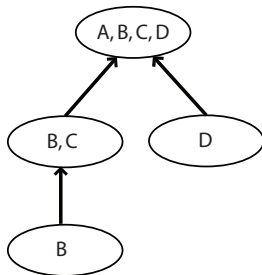
# Max-tree (or min-tree) representation



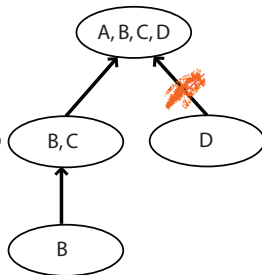
Original Image



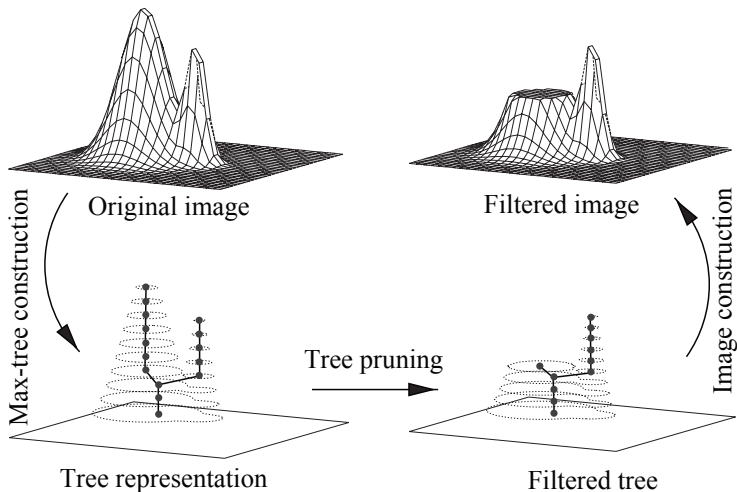
Original Image



Max-Tree



Max-Tree



from (Salembier ITIP 00)

# Attribute opening



Original  
Image



Area  
Opening

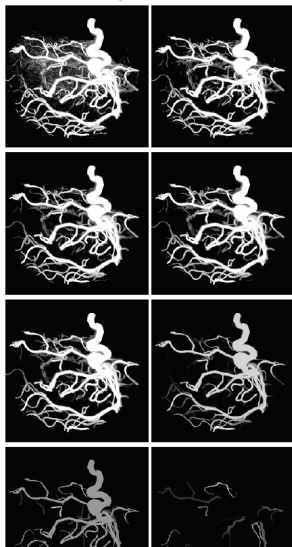


Moment of Inertia  
Opening

from (Wilkinson ISMM 00)

# Examples

Filtering depending on elongation (Meijster, 2002) :



Entropy criterion (Salembier, 1998) :

Original



Entropy Operator



from (Salembier ITIP 98)

Movement analysis (Salembier, 1998) :



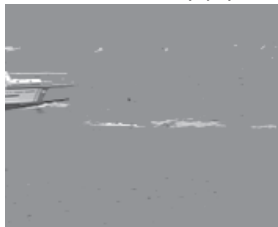
Original frame



Objects with  
translation (0,0)

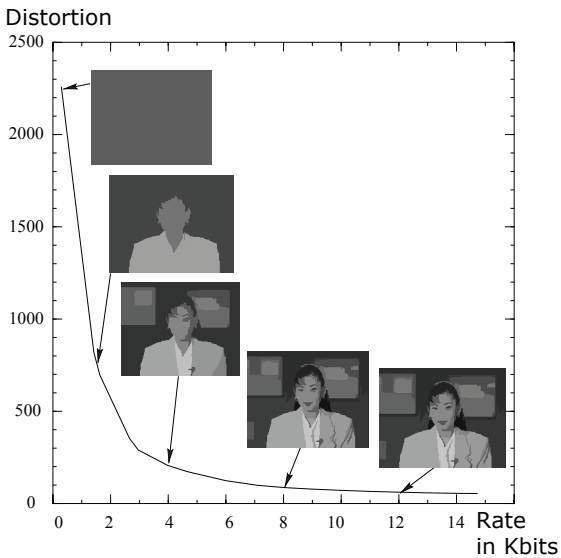


Objects with  
translation (2,0)



Remaining Objects

## Compression (Salembier, 2000) :

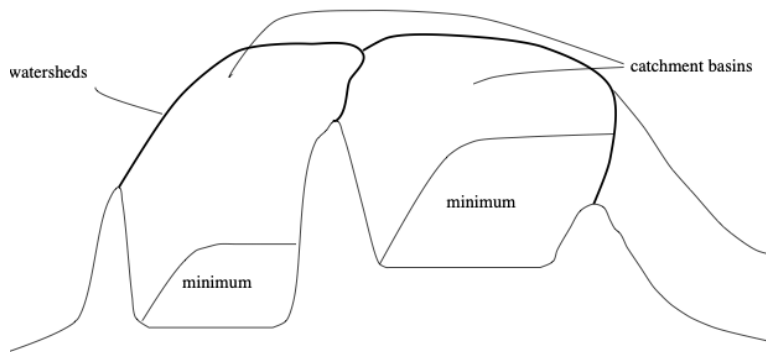


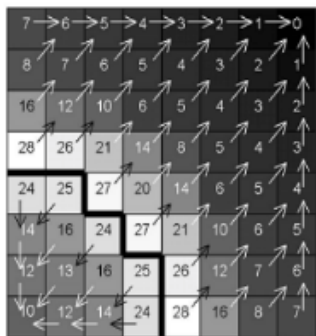
from (Salembier ITIP 00)

- Criteria for image segmentation:
  - simplicity
  - regularity
  - fidelity to the data
  
- Two morphological paradigms:
  - flat zones (using connected filters)
  - catchment basins and watersheds

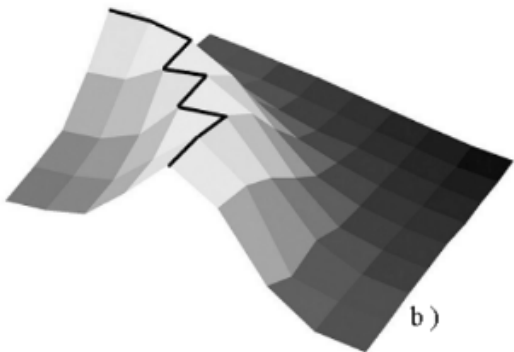
# Watersheds





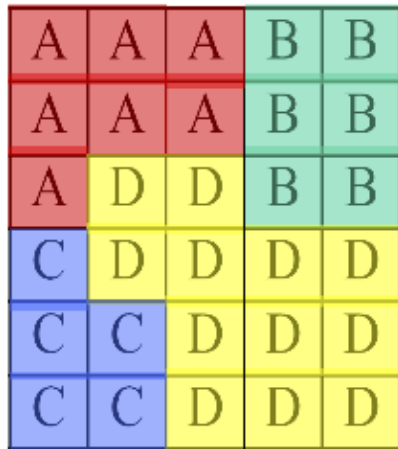
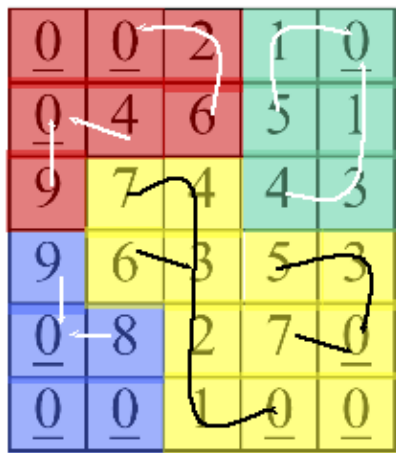


a)



b)

## Watersheds based on local conditions



Several solutions...

# Watersheds: definition

Steepest descent:

$$Desc(x) = \max\left\{\frac{f(x) - f(y)}{d(x, y)}, y \in V(x)\right\}$$

Ramp of a path  $\pi = (x_0, \dots, x_n)$ :

$$T_f(\pi) = \sum_{i=1}^n d(x_{i-1}, x_i) Cost(x_{i-1}, x_i)$$

with

$$Cost(x, y) = \begin{cases} Desc(x) & \text{if } f(x) > f(y) \\ Desc(y) & \text{if } f(y) > f(x) \\ (Desc(x) + Desc(y))/2 & \text{if } f(x) = f(y) \end{cases}$$

## Topographic distance

$$T_f(x, y) = \inf\{T_f(\pi), \pi = (x_0 = x, x_1, \dots, x_n = y)\}$$

(equals 0 on a plateau)

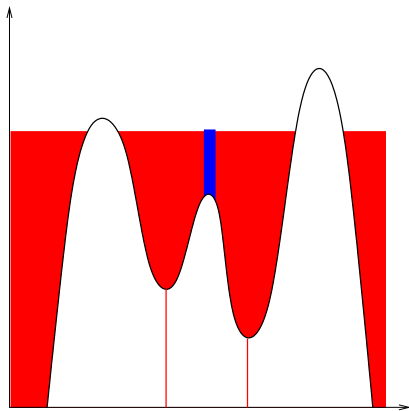
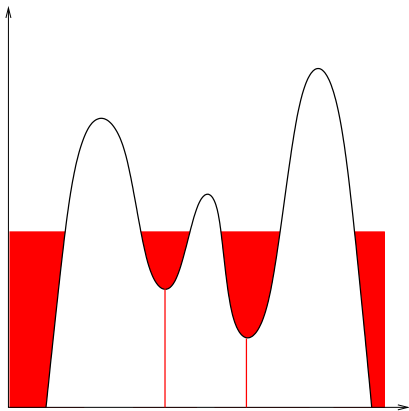
**Catchment basin** associated with the regional minimum  $M_i$ :

$$CB(M_i) = \{x \mid \forall j \neq i, T_f(x, M_i) + f(M_i) < T_f(x, M_j) + f(M_j)\}$$

**Watersheds:**

$$WS(f) = [\cup_i CB(M_i)]^C$$

## Approach by immersion



# Construction of the watersheds

$f$  such that  $f(x) \in [h_{\min}, h_{\max}]$ ,  $f^h = \{x, f(x) \leq h\}$

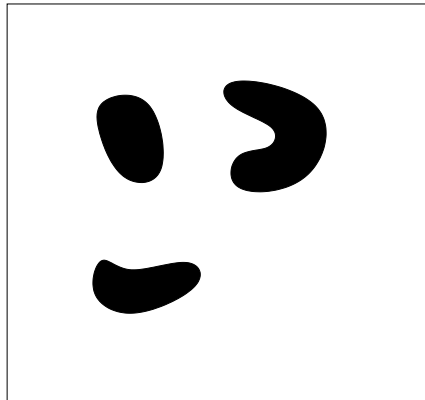
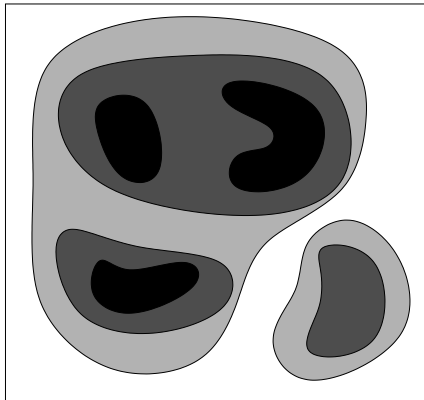
$$X_{h_{\min}} = f^{h_{\min}}$$

$$X_{h+1} = \text{MinReg}_{h+1}(f) \cup \text{ZI}_{f^{h+1}}(X_h)$$

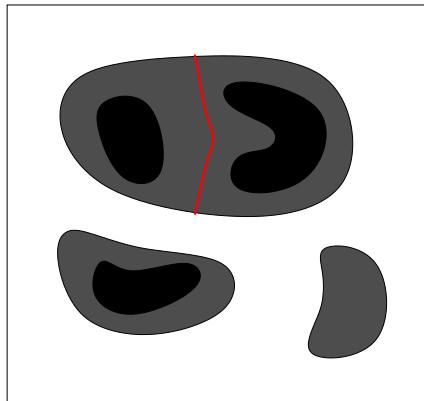
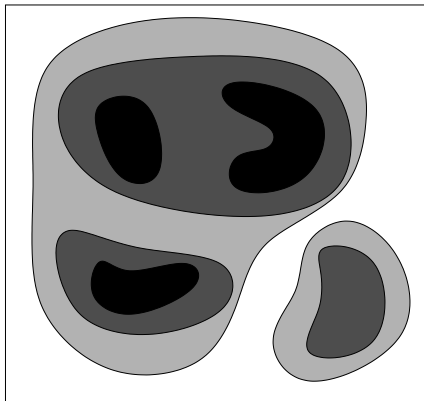
$$CB = X_{h_{\max}}$$

$$WS(f) = X_{h_{\max}}^C$$

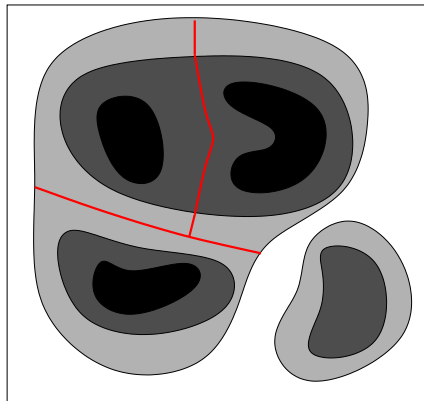
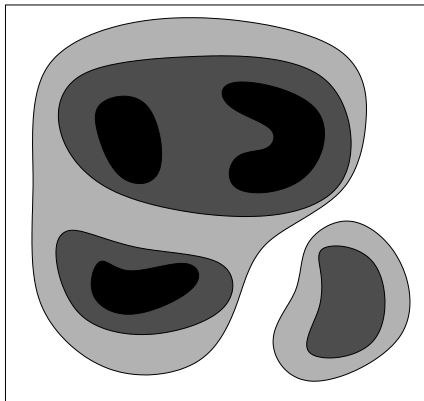
# Illustration of the algorithm



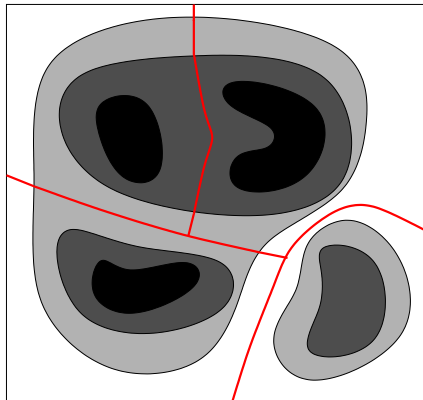
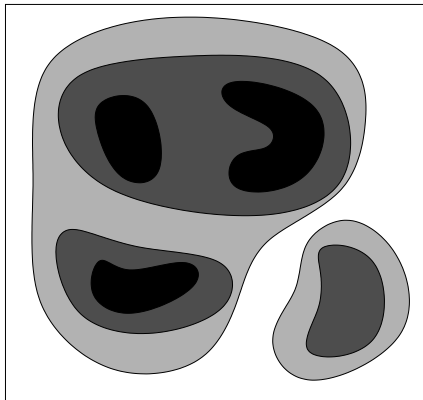
# Illustration of the algorithm



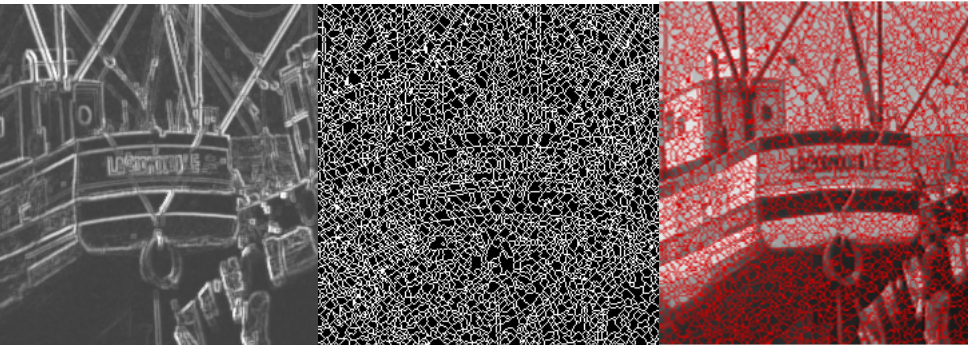
# Illustration of the algorithm

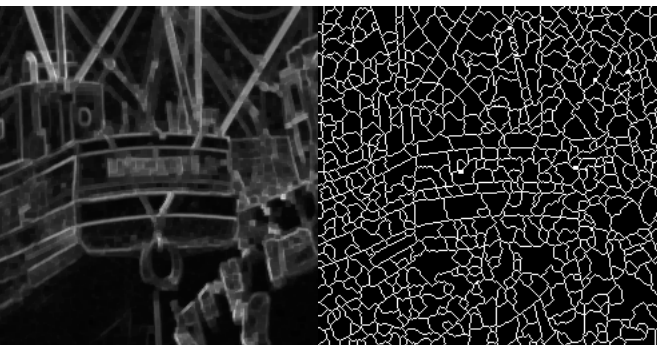


# Illustration of the algorithm

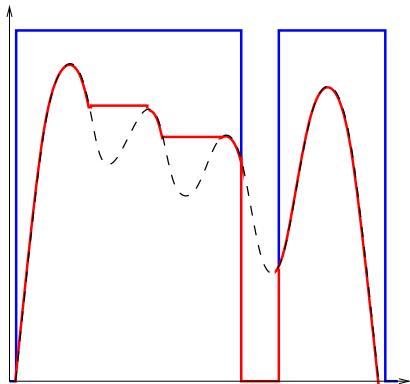
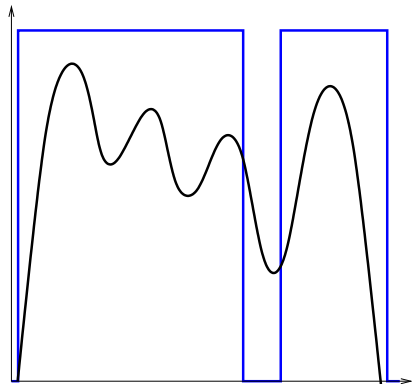


# Watersheds and oversegmentation





# Geodesic erosion in order to impose markers



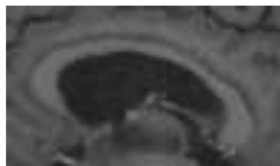
## Watersheds constraint by markers

$f$ : function on which watersheds should be applied

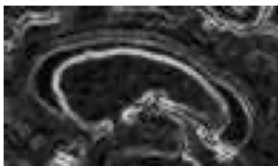
$g$ : marker function (selects regional minima)

Reconstruction:  $E_{f \wedge g}(g, B_\infty)$  (only the selected minima)





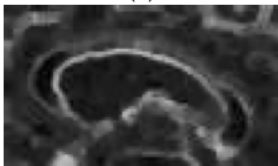
(a)



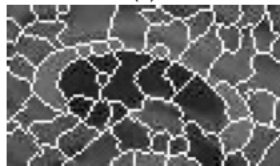
(b)



(c)

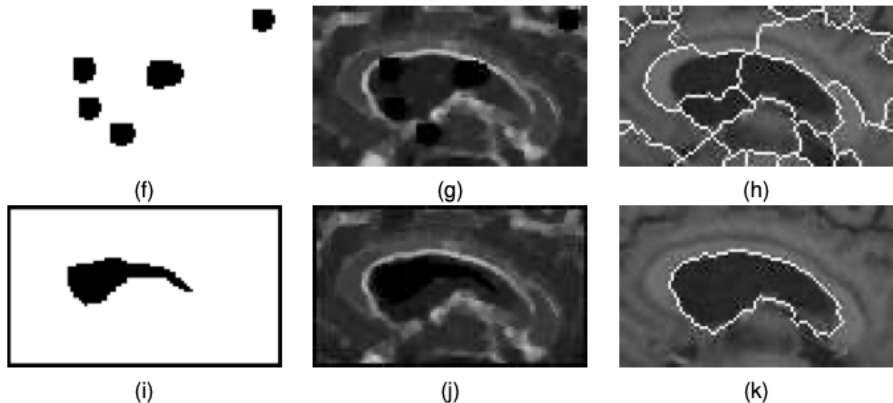


(d)



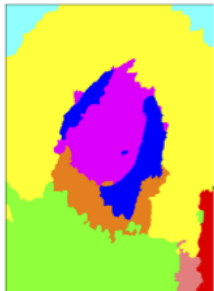
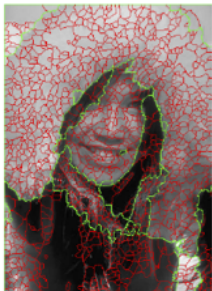
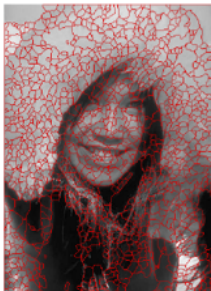
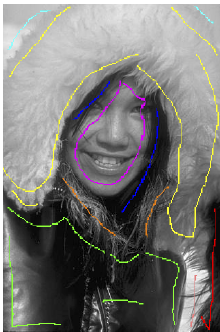
(e)

(a) Original image (from a brain MRI). (b) Morphological gradient. (c) Watersheds. (d) Closing (size 1) of the gradient. (e) Watersheds applied on the closing of the gradient.

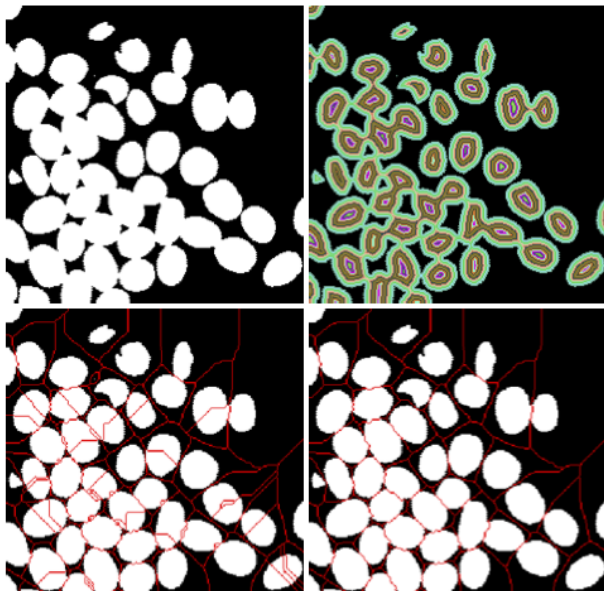


(f) Markers from regional minima. (g) Reconstruction of the gradient.  
 (h) Watersheds. (i) Markers *inside* the ventricles and on the image border.  
 (j) Reconstructed gradient. (k) Watersheds providing the right contours of the ventricles.

## Interactive marker-based segmentation



# Separation of connected binary objects



# Watershed as an energy minimization problem

Minimum of the energy defined as (Boomgard, 2000):

$$E = \sum_i \int \int_{D_i} (f(D_i) + T_f(x, D_i)) dx$$

$D_i$  = regional minimum

$f(D_i)$  = value of the regional minimum

- using closing
- watersnakes (Boomgard, 2003): additional term controlling the length of the contours

$$E = \sum_i \left( \int \int_{D_i} (f(D_i) + T_f(x, D_i)) dx + \beta \int_{\partial D_i} ds \right)$$

- geometrical constraints
- viscous flooding (Vachier and Meyer)

## Watersnakes: examples (Boomgard, 2003)

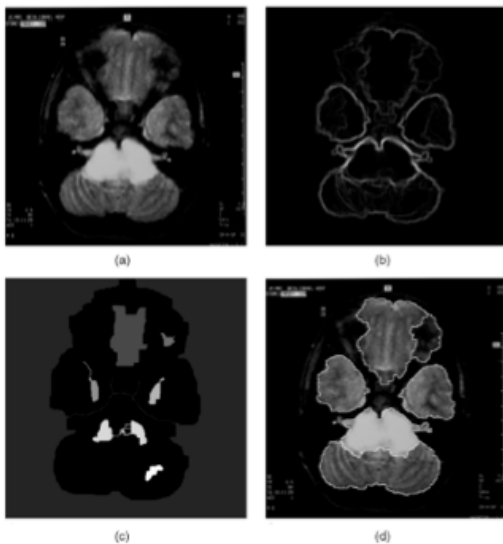


Fig. 6. (a) A brain image. (b) The relief computed from morphological gradient. (c) The markers extracted. (d) The result of the original watershed segmentation, shown for comparison.

## Watersnakes: examples (Boomgard, 2003)

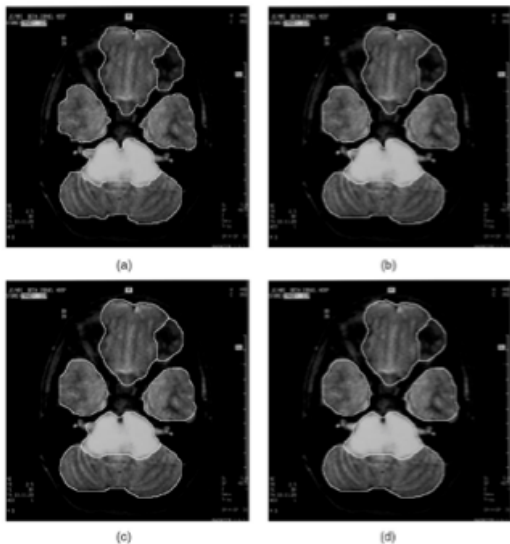


Fig. 7. The segmentation result of the watersnake algorithm based on energy discretization with (a)  $\beta = 10$ , (b)  $\beta = 50$ , (c)  $\beta = 100$ , and (d)  $\beta = 150$ . Note, in comparison with the original watershed segmentation in Fig. 6d, that the results in figure are smoother, but still identify the main objects.

## Watersnakes: examples (Boomgard, 2003)

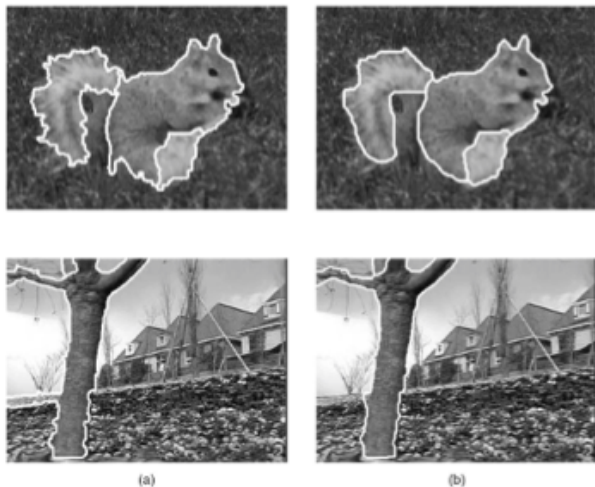
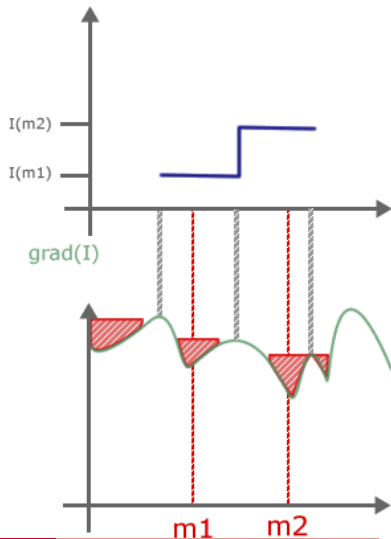


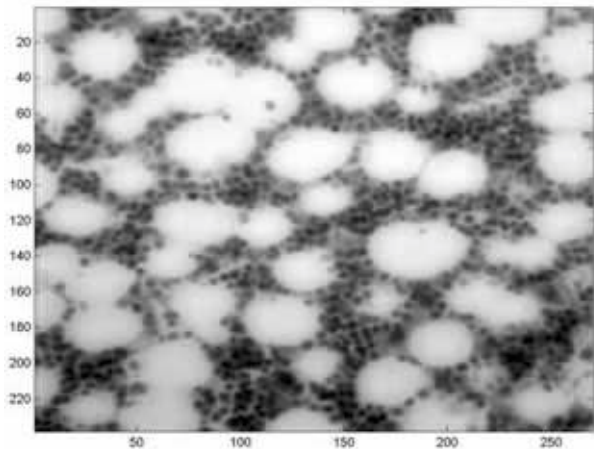
Fig. 8. Segmentation by (a) watershed and (b) watersnake ( $\beta = 50$ ). In the bottom row, the result is shown for the object of interest only.

## Hierarchical watersheds

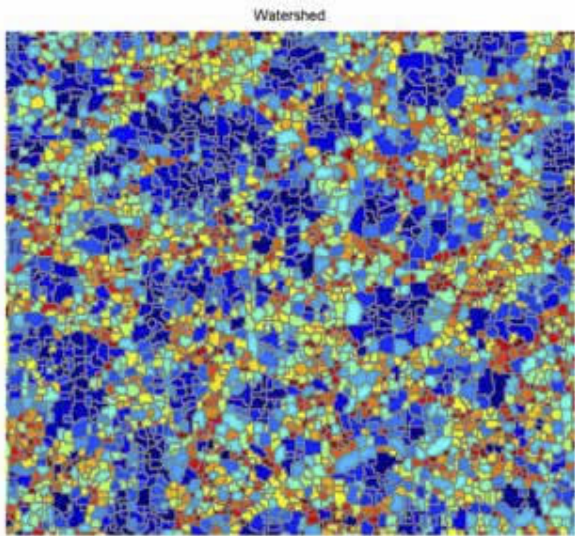
Iterative filling of catchment basins:



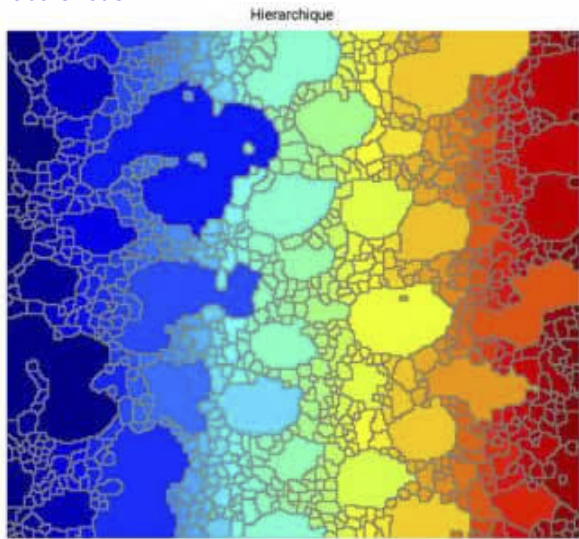
## Hierarchical watersheds



## Hierarchical watersheds



## Hierarchical watersheds



# Hit-or-Miss Transformation (HMT)

Structuring element:  $T = (T_1, T_2)$ , with  $T_1 \cap T_2 = \emptyset$

HMT:

$$X \otimes T = E(X, T_1) \cap E(X^C, T_2)$$

Thinning (if  $O \in T_1$ ):

$$X \circ T = X \setminus X \otimes T$$

Thickening (if  $O \in T_2$ ):

$$X \odot T = X \cup X \otimes T$$

For  $T' = (T_2, T_1)$ :

$$X \circ T = (X^C \odot T')^C$$

- end points
- multiple points
- convex hull
- homotopic skeleton
- ...

- And many other operations!
- Current trend: mathematical morphology associated with artificial neural networks
  - additional input
  - post-processing
  - non-linear convolutions
  - inside the cost (loss) function to be optimized
  - ...