

Digital Representations

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- Tessellations
- Digital topology
- Representation of geometrical entities
- Distance function

Digital representation of images

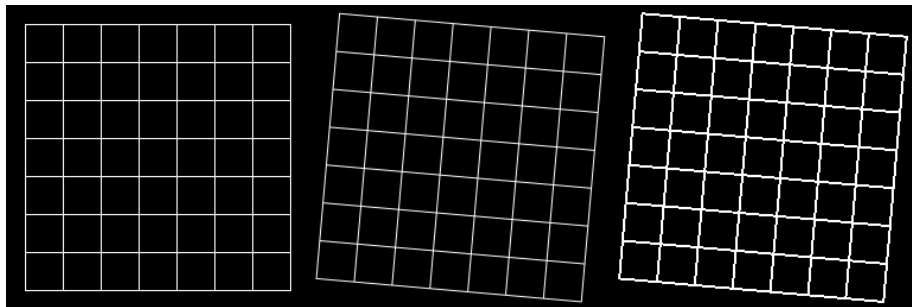
Digital image:

- representation as an array
- sampling from \mathbb{R}^2 or \mathbb{R}^3 into \mathbb{Z}^2 or \mathbb{Z}^3

Two possible approaches to process points in \mathbb{Z}^n :

- embed \mathbb{Z}^n in \mathbb{R}^n , then apply operations and transformations in the continuous space
- definition of operations and transformations directly in the digital space
 - definitions?
 - preservation of the expected effects?
 - preservation of the properties?

A simple example: rotation



$$(x', y') = R(x, y)$$

Issues:

- $(x, y) = \text{integer coordinates} \Rightarrow (x', y') ?$
- Computation?
- Properties?

$$\pi/4 \text{ rotation: } x' = (x - y) \frac{\sqrt{2}}{2} \quad y' = (x + y) \frac{\sqrt{2}}{2}$$

	a	b	c	
	d	e	f	
	g	h	i	

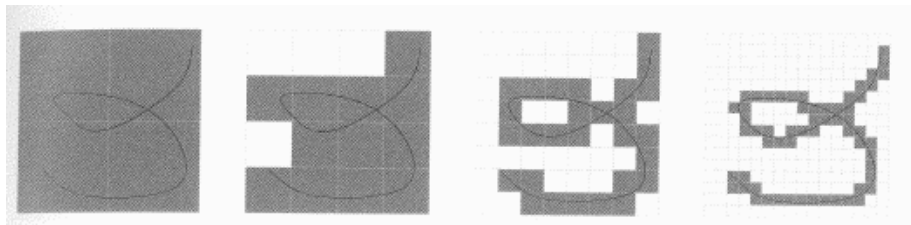
	c			
b		f		
ad	e	h	i	
	g			

Direct transformation

	c			
b	e	f		
d	e	h		
	g			

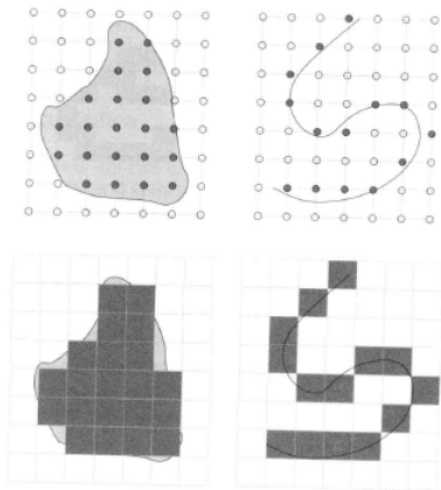
Inverse transformation
(closest point interpolation)

Topology and resolution



Source: D. Cœurjolly, A. Montanvert, J. M. Chassery (2007)

Curve digitization



Source : D. Cœurjolly, A. Montanvert, J. M. Chassery (2007)

Tessellation = partition of the continuous space \mathbb{R}^n into elementary cells

Constraints:

- physical sensors (regularity)
- usage of the representation (regularity, simplicity)

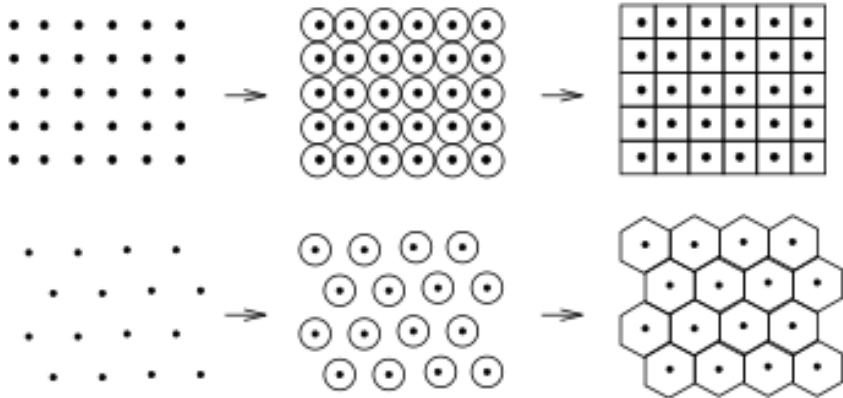
1 Tessellation from point distribution

- distribution of points P
 - regular \Rightarrow classical grids
 - irregular \Rightarrow Voronoï diagram
- attribution of a cell V_P to each point

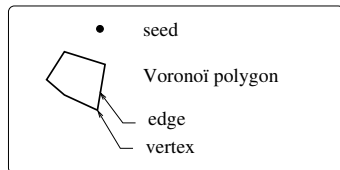
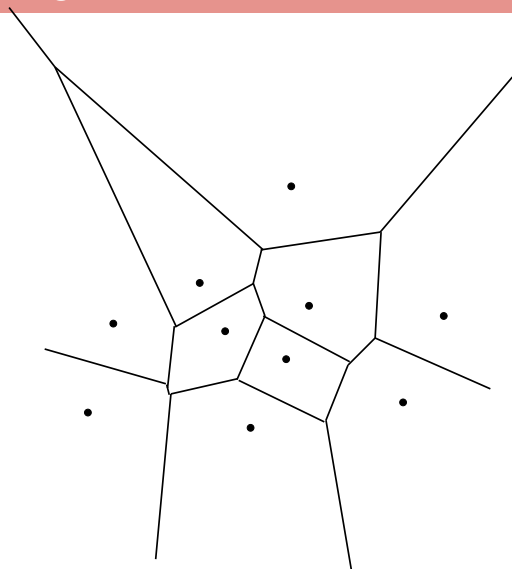
2 Tessellation from cell juxtaposition

- prior definition of a cell model V_P
- juxtaposition of V_P so as to build a partition
- constraints:
 - V_P convex and regular
 - vertices in contact with other vertices only

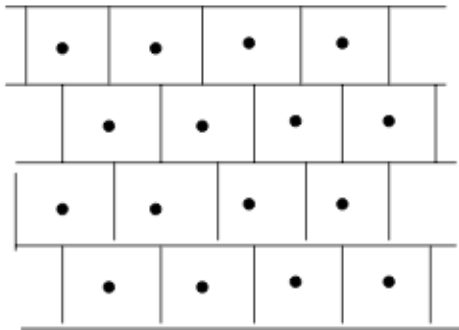
Regular distributions



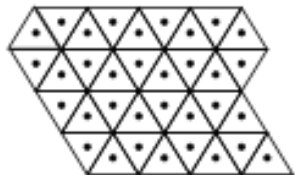
Irregular distributions: Voronoï diagram



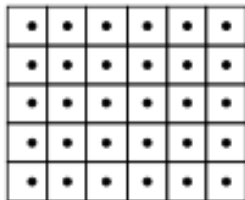
An excluded configuration



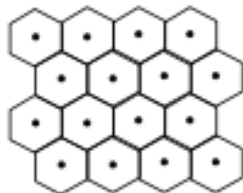
Regular tessellations of the plane



triangular



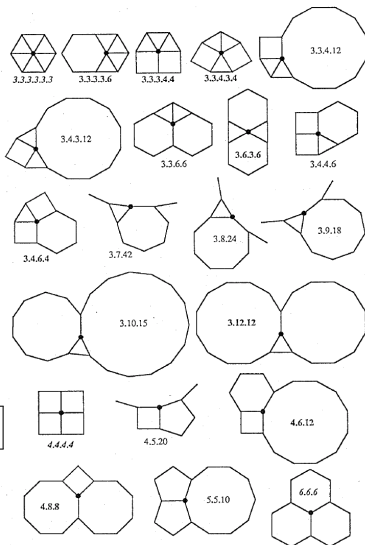
square



hexagonal

Semi-regular tessellations

regular



regular

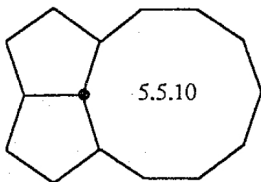
regular

Examples

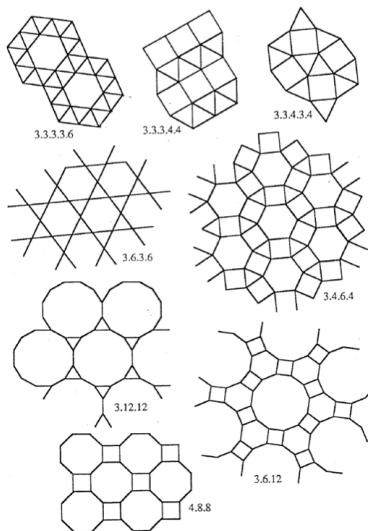
Admissible:



Non-admissible:



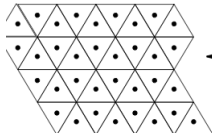
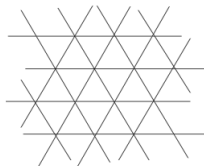
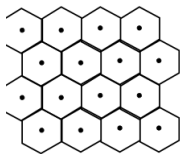
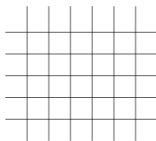
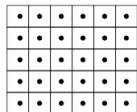
Admissible semi-regular tessellations



A complex tessellations (Escher)...

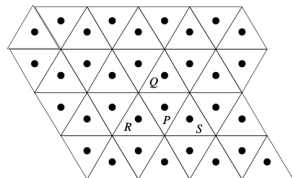
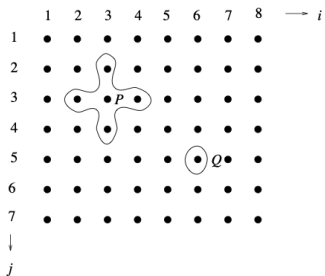


Duality between tessellations and mesh (or grid)

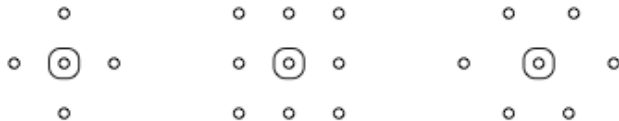


- Classical topology in a countable set of points:
 - every point is an open set of the topology
 - not well adapted to the representation of connected sets
- Direct definition of a topological basis
 - possible on triangular and square tessellations, not on hexagonal tessellations
 - depends on point localization
 - does not satisfy Jordan theorem
- Direct definition of elementary neighborhood
 - digital connectivity
 - image = graph

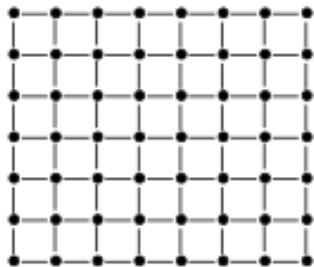
Direct definition of a topological basis: examples



Elementary neighborhood

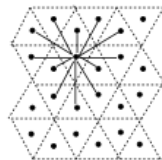
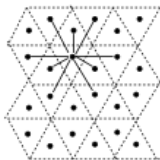
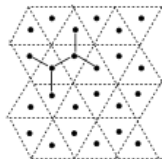
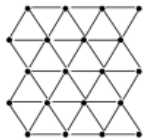
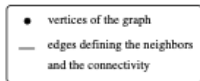
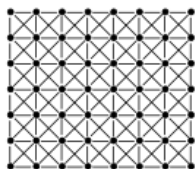


4-connectivity graph

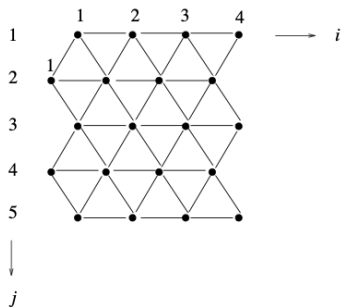


- vertices of the graph
- edges defining the neighbors and the connectivity

Different grids and associated connectivities

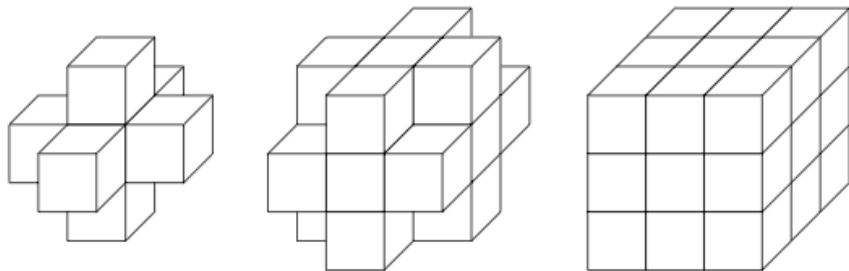


Neighbor coordinates on a hexagonal grid



- for j even: $(i - 1, j - 1)$, $(i, j - 1)$, $(i - 1, j)$, $(i + 1, j)$, $(i - 1, j + 1)$, $(i, j + 1)$,
- for j odd: $(i, j - 1)$, $(i + 1, j - 1)$, $(i - 1, j)$, $(i + 1, j)$, $(i, j + 1)$, $(i + 1, j + 1)$.

Elementary neighborhood on a 3D cubic grid



Path and connected component

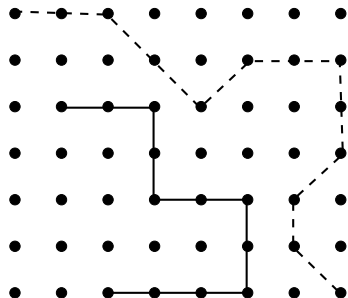
- **4-connected path** = sequence of points $(i_k, j_k)_{1 \leq k \leq n}$ such that:

$$\forall k, 1 \leq k < n, |i_k - i_{k+1}| + |j_k - j_{k+1}| \leq 1$$

- **8-connected path** = sequence of points $(i_k, j_k)_{1 \leq k \leq n}$ such that:

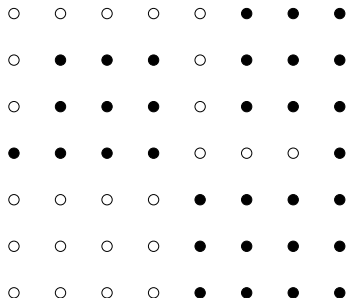
$$\forall k, 1 \leq k < n, \max(|i_k - i_{k+1}|, |j_k - j_{k+1}|) \leq 1$$

- **4-connected component** = set of points \mathcal{S} such that for any (P, Q) in \mathcal{S} , there exists a 4-connected path from P to Q , included in \mathcal{S} , and maximal for this property.
- **8-connected component** = set of points \mathcal{S} such that for any (P, Q) in \mathcal{S} , there exists a 8-connected path from P to Q , included in \mathcal{S} , and maximal for this property.



—— 4-connected path

- - - 8-connected path



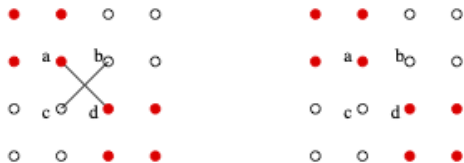
○ background

● objects

two 4-connected components

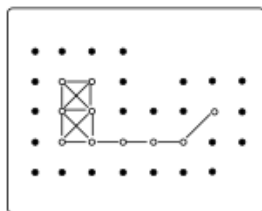
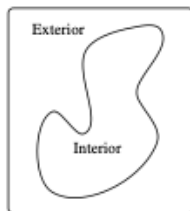
one 8-connected component

Topological paradox



Jordan theorem

- Continuous case: any simple and closed curve divides the space into two connected components, one inside the curve and one outside.
- Digital case: duality between 4-connectivity and 8-connectivity on a square grid
 - 4-connected curve \Leftrightarrow 8-connected background,
 - 8-connected curve \Leftrightarrow 4-connected background.
- Digital case on a hexagonal grid: 6-connectivity for both objects and background (no topological problem).
- Extension to 3D.



Some definitions in the digital case

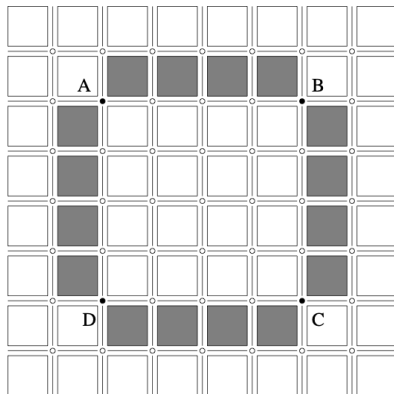
- **Simple and closed 4-connected path:** 4-connected path (A_0, \dots, A_n) such that $n \geq 4$, $A_i = A_j$ iff $i = j$, and A_i 4-neighbor of A_j iff $i = j \pm 1[n + 1]$
- **Horizontal half-line** from $M = (a, b)$:

$$H_M = \{(a + k, b), k = 0, 1, 2, \dots\}$$

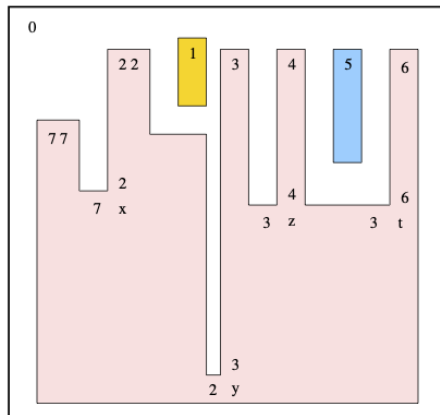
- **Inside A :** set of points M such that H_M crosses A an odd number of times.
- **Outside A :** set of points M such that H_M crosses A an even number of times.

⇒ proof of the digital version of Jordan theorem.

Cellular complexes



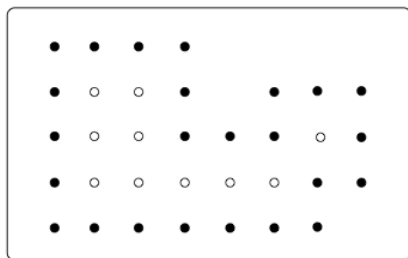
Connected component labeling



initial label	pointer	final label
0		0
1		1
2		2
3 →	2	2
4 →	3	2
5		3
6 →	3	2
7 →	2	2

Example of topological characteristic: Euler number

- Number of connected components N_{cc}
- Number of holes N_t
- Euler number $E = N_{cc} - N_t$



- Object
- Hole

- 8-connected objects and 4-connected holes: $N_{cc} = 1$ and $N_t = 2$, hence $E = -1$
- 4-connected objects and 8-connected holes: $N_{cc} = 1$ and $N_t = 1$, hence $E = 0$

v	•
e	• • • •
d	• • • •
t	• • • • • • • • • • • •
q	• • • •

- 8-connected objects and 4-connected holes:

$$E = v - e - d + t - q$$

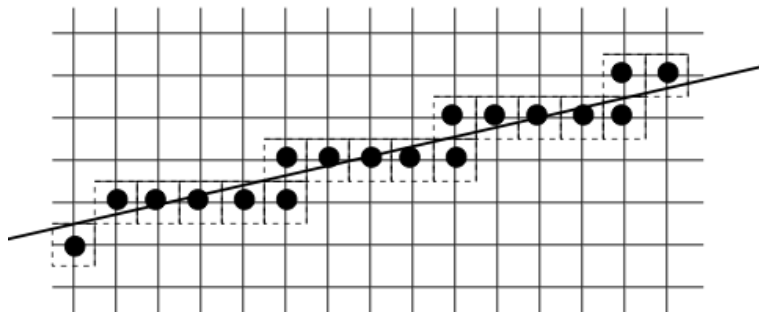
- 4-connected objects and 8-connected holes:

$$E = v - e + q$$

- How to go from the continuous domain to the digital one, and vice-versa?
- How to represent a geometric entity on a digital grid, while preserving its properties?
- Which are the continuous representations of a discrete one?
- Which are the exact intersections of a continuous representation and the digital grid?

Example of straight lines or segments

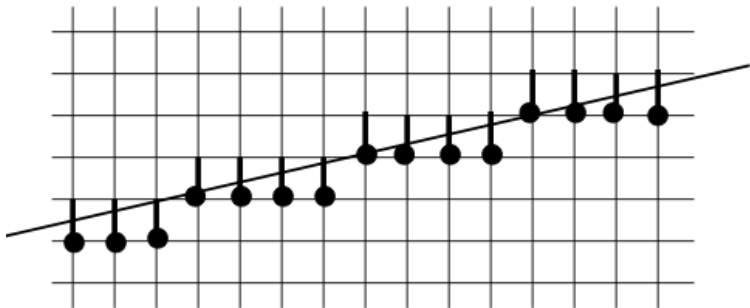
Digitization of a continuous line



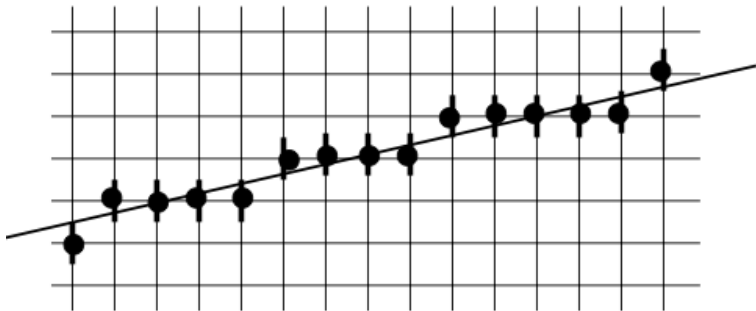
Semi-open square



Digital representation of the continuous line



● Digital representation of the continuous line



● Digital representation of the continuous line

Cf Bresenham algorithm

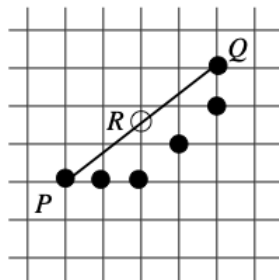
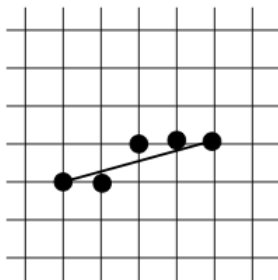
Characterization of a digital straight line segment

Cord property

\mathcal{S} satisfies the cord property iff:

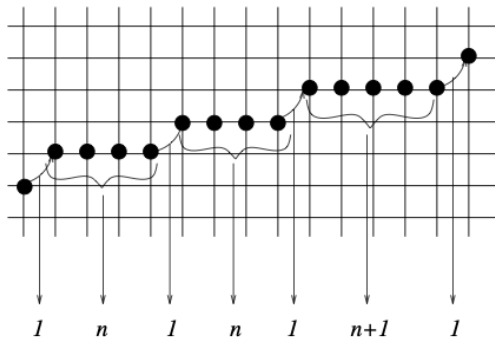
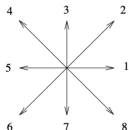
$$\forall (P, Q) \in \mathcal{S}, \forall R \in [P, Q], \exists T \in \mathcal{S}, d_{\infty}(T, R) < 1$$

with $d_{\infty}((x, y), (x', y')) = \max(|x - x'|, |y - y'|)$



Syntactic characterization

- only two “neighbor” directions
- for one direction: sections of length 1
- for the other direction: sections of length n or $n + 1$



Analytical digital straight lines

$$y = ax + b$$

Intersections with the grid?

- Condition for non-empty intersection:

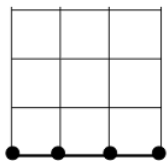
$$a = \frac{p}{q}$$

p and q integers, co-prime, and:

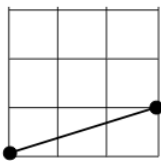
$$p \leq q \leq N$$

- Farey sequence:
 - image of size $N \times N$ and slope less than 1
 - \Rightarrow possible slopes = Farey sequence of order N : $F(N)$ (cardinality approximately $3N^2/\pi^2$)
 - recursive construction ($\frac{m+m'}{n+n'}$ between $\frac{m}{n}$ and $\frac{m'}{n'}$)

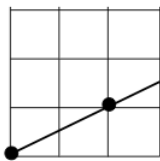
Example for $N = 4$ ($a \leq 1$) : $F(N) = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$



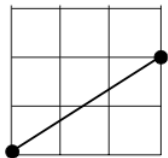
$p/q = 0$



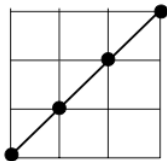
$p/q = 1/3$



$p/q = 1/2$



$p/q = 2/3$

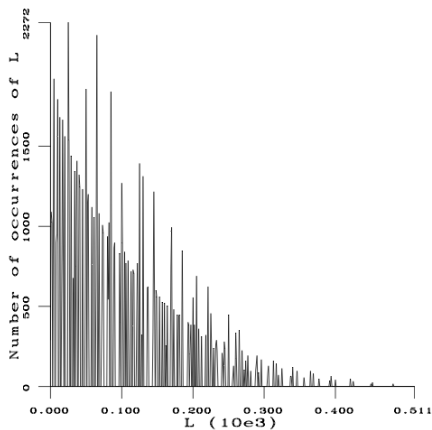


$p/q = 1$

Length of a digital straight segment

$$a^2 + b^2 = L$$

with a and b integer

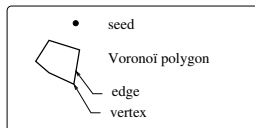
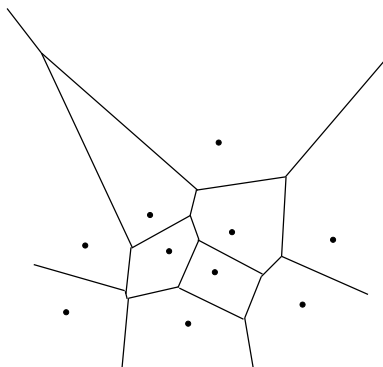


Voronoi diagram

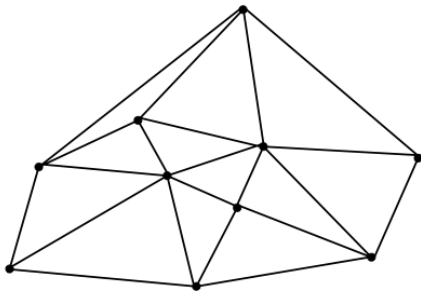
- Useful representation for shapes, image structures...
- Seeds $\{P_1, P_2, \dots, P_n\}$
- Voronoi cells:

$$V(P_i) = \{P \in \mathbb{R}^2 \mid \forall j, 1 \leq j \leq n, d(P, P_i) \leq d(P, P_j)\}$$

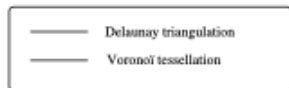
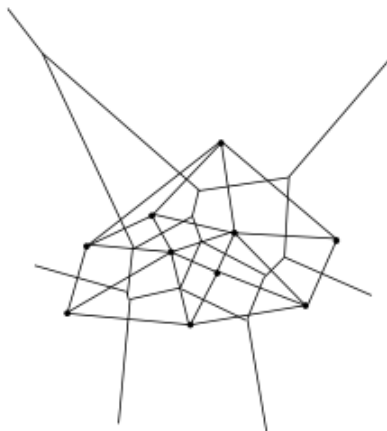
- For the Euclidean distance: $V(P_i) = \text{convex polygon}$



Delaunay triangulation

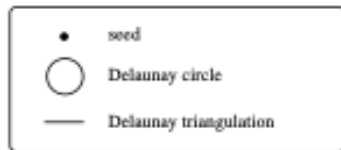
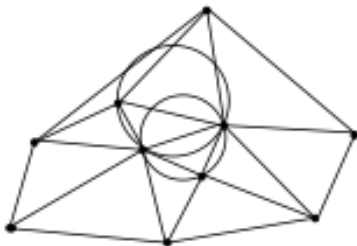


Duality

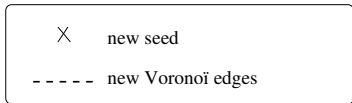
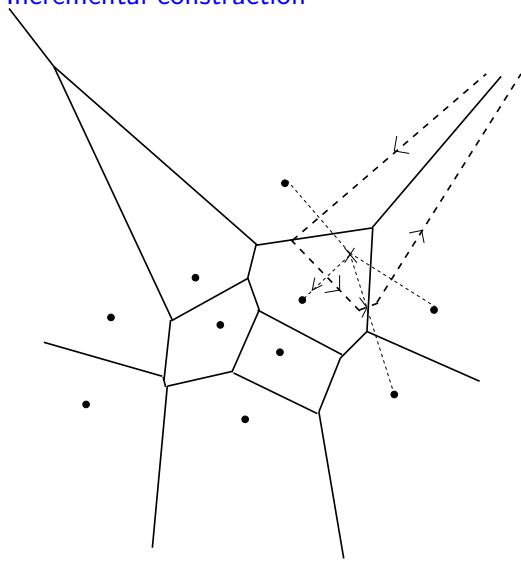


Properties:

- if there are no 4 co-circular points, every Voronoï vertex is equidistant of exactly 3 seeds
- any Voronoï vertex is the center of a circle (called Delaunay circle) passing through 3 seeds and containing no other seed
- $V(P_i)$ is non-bounded iff P_i belongs to the convex hull of the P_j s



Incremental construction



A few geometric applications:

- minimal distance between two sets of points
- triangulation such that the circle circumscribed to each triangle is empty
- convex hull of a set of points
- ...

Discrete distances

- $\mathcal{P} = \{\vec{p}_1, \dots, \vec{p}_m\}$ set of vectors generating a graph
- Associated length d_i
- Conditions:
 - $\vec{p}_i \in \mathcal{P} \Rightarrow -\vec{p}_i \in \mathcal{P}$
 - $\vec{p}_i \in \mathcal{P}, \lambda \vec{p}_i \in \mathcal{P} \Rightarrow \lambda = \pm 1$
 - $\|\vec{p}_i\| = \|\vec{p}_j\| \Rightarrow d_i = d_j$

Distance between to vertices / points x and y :

$$d(x, y) = \frac{1}{s} \min \left\{ \sum_{i=1}^m n_i d_i \mid n_i \in \mathbb{N}, \sum_{i=1}^m n_i \vec{p}_i = \vec{xy} \right\}$$

s : scale factor

Distance function

Binary image with objects $O \rightarrow$ distance map image where the value at x is $d(x, O) = \min_{y \in O} d(x, y)$

- global concept \Rightarrow local computation by propagating local distances
- requirements:
 - good approximation of the Euclidean distance
 - fast algorithms

Masks representing local distances

									11		11					
	1			1	1	1		4	3	4		11	7	5	7	11
1	0	1		1	0	1		3	0	3			5	0	5	
	1			1	1	1		4	3	4		11	7	5	7	11
													11		11	
	a				b				c					d		

Parallel algorithm

- f^k : image at iteration k
- g : mask
- f^0 : points of objects set to 0, points of the background set to $+\infty$

$$f^k(x) = \min\{f^{k-1}(y - x) + g(y), y \in \text{support}(g)\}$$

- number of iterations: depends on image size, object size, shape...
- two images in memory
- can be adapted for any grid (2D or 3D) and any mask
- can be parallelized

Sequential algorithm

- two scans of the image, in opposite directions
- masks g_1 and g_2 containing the points already examined according to the scan direction (+ origin)
- f^0 : points of objects set to 0, points of the background set to $+\infty$
- for $k = 1, 2$

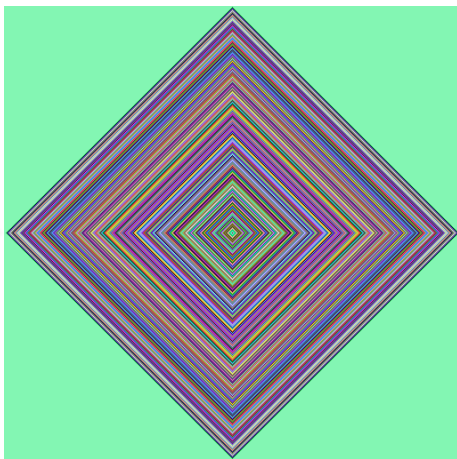
$$f^k(x) = \min\{f^{k-1}(x), f^k(y - x) + g_k(y), y \in \text{support}(g_k)\}$$

- fast algorithm
- only one image in memory
- can be adapted for any grid (2D or 3D) and any mask
- recursive

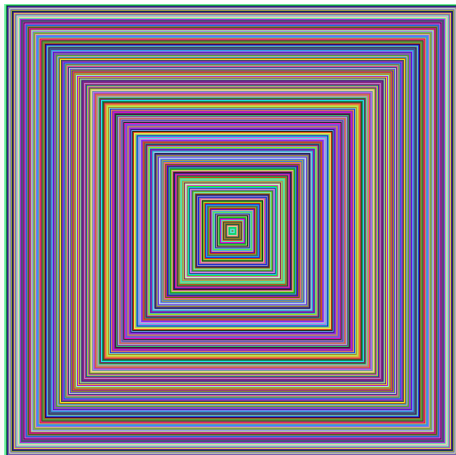
Algorithms based on object contours

- Using chains:
 - contour chaining
 - point displacement and rewriting rules
 - adjustments
- Using queues
 - FIFO initialized with contour points
 - for each point of the queue: computation of the neighbors, distance value increment, and neighbors added in the queue
 - applies in 3D as well

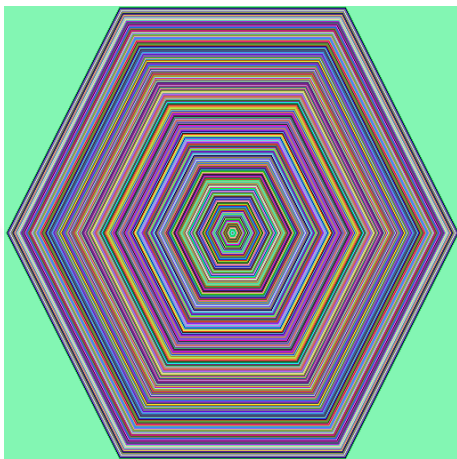
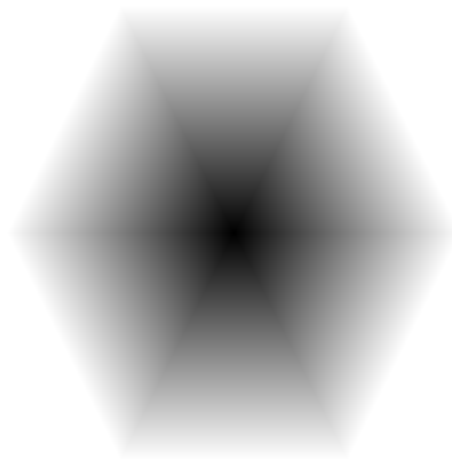
Distance map (4-connectivity mask)



Distance map (8-connectivity mask)



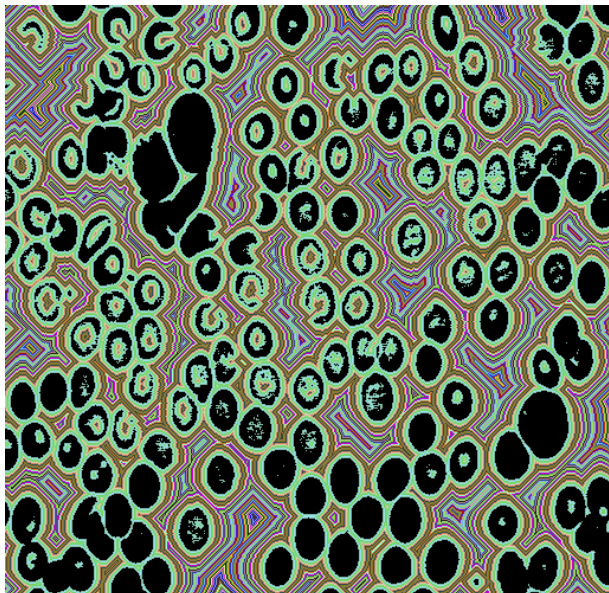
Distance map (6-connectivity mask)

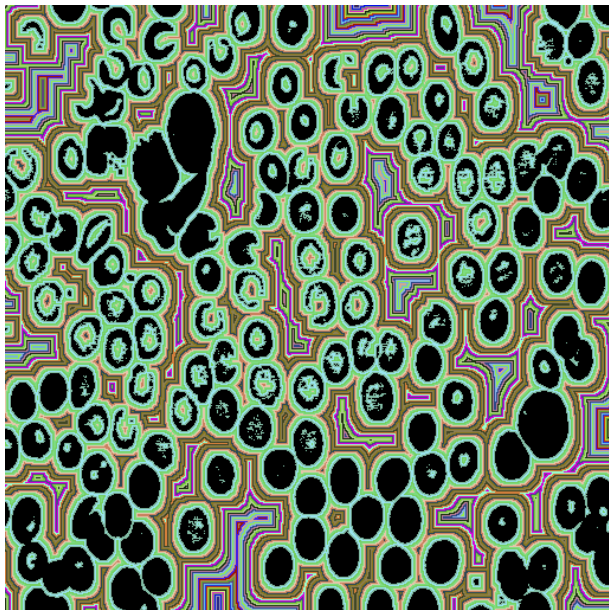


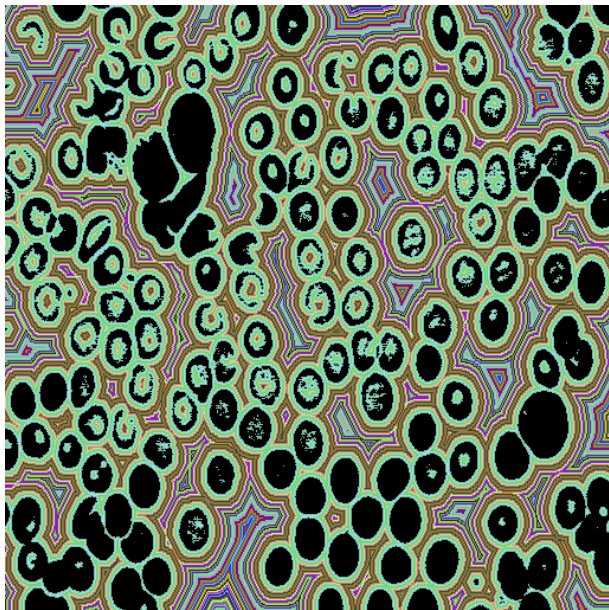
Comparison 4c / 8c / 6c / 5-7-11

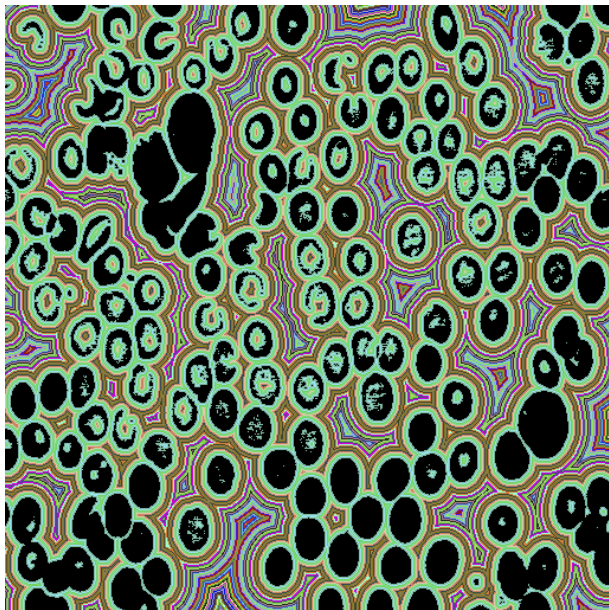


Example on a binarized biological image

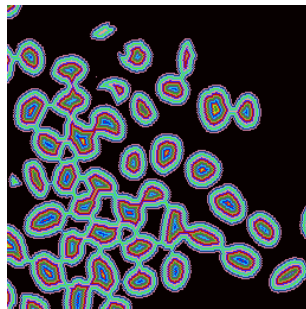
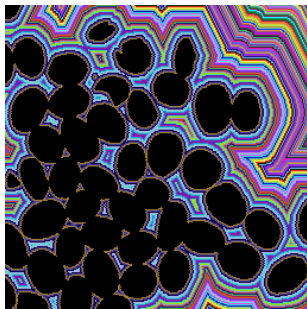
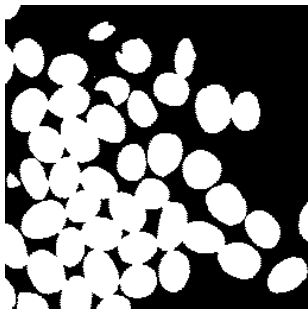




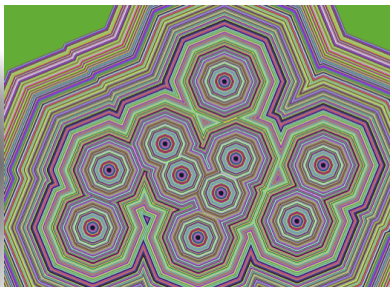
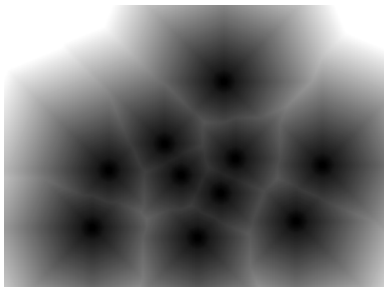
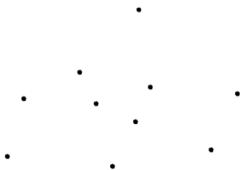


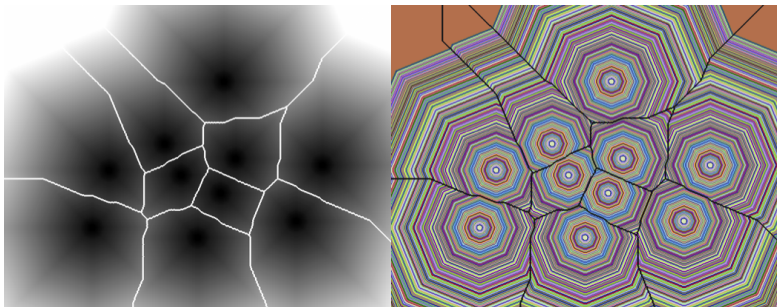


Example on a coffee bean image



Voronoi diagram from a discrete distance





- Distance computation (e.g. model-based object recognition, scene understanding)
- Registration
- Mathematical morphology operations on binary images

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