Tomographic Reconstruction

Isabelle Bloch

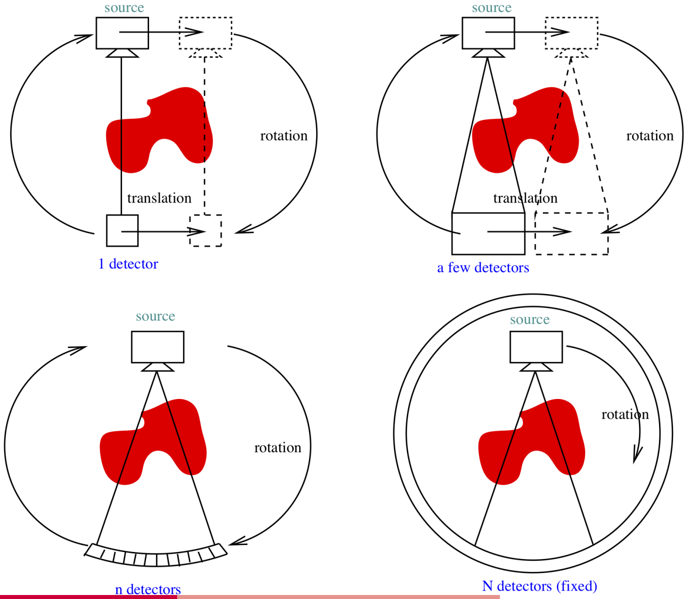
LIP6, Sorbonne Université

isabelle.bloch@sorbonne-universite.fr

Content

- Principle of tomography
- Backprojection
- Analytical methods
- Algebraic methods
- Regularization
- Extensions

CT acquisition systems

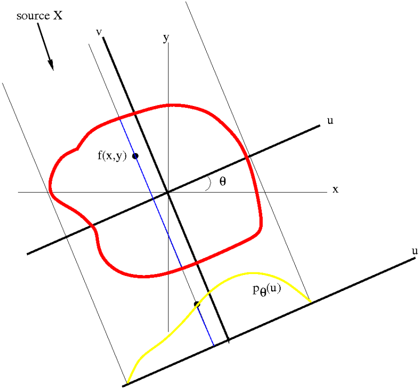


Attenuation for a monochromatic X-ray beam:

$$I = I_0 \exp(-\int_{-\infty}^{+\infty} f dv)$$

$f(x, y)$ = attenuation at point (x, y) = function to be reconstructed

Acquisition of projections



Other modalities

- nuclear imaging (SPECT, PET)
- electric impedance tomography
- ...

Different physical principles - Similar reconstruction problems.

Radon transform

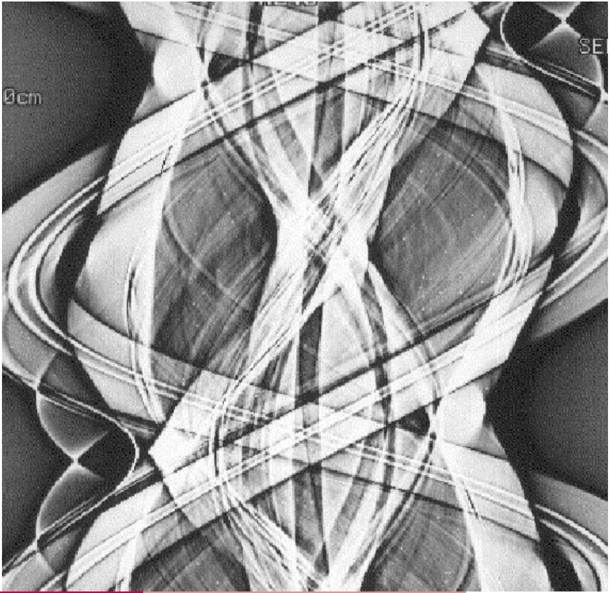
$$\begin{aligned} R[f](u,\theta) &= p_\theta(u) \\ &= \int_{D_\theta} f(u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta)dv \end{aligned}$$

Note that $p_\theta(u) = p_{\theta+\pi}(-u)$

Reconstruction:

$$\boxed{\{p_\theta(u),\ \theta \in [0,\pi[,\ u \in \mathbb{R}\} \ \rightarrow\ \{f(x,y),\ (x,y) \in \mathbb{R}^2\}}$$

Sinogram



Backprojection

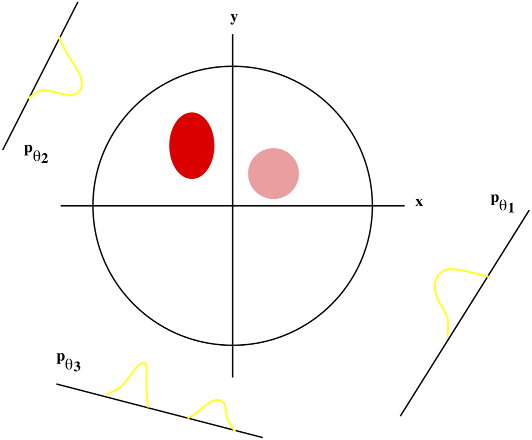
- of a projection :

$$h_\theta(x, y) = p_\theta(x \cos\theta + y \sin\theta)$$

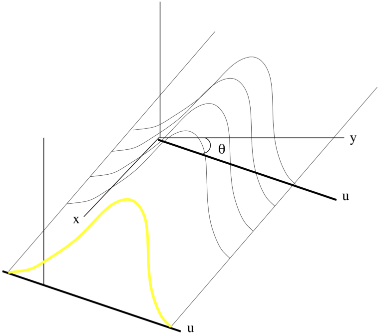
(value at (x, y) of the projection of angle θ
at point on which (x, y) projects)

- of all projections:

$$B[p](x, y) = \int_0^\pi p_\theta(x \cos\theta + y \sin\theta) d\theta$$



y
x
pθ2
pθ1
pθ3



y
θ
u
x
u

Inversion - 1

Projection theorem

$$FT[p_\theta](U) = FT[f](U\cos\theta, U\sin\theta)$$

(FT = Fourier transform)

$\Rightarrow$ Reconstruction scheme:

$$\boxed{\begin{array}{c} \{p_\theta(u)\} \\ \Downarrow \\ \{FT[p_\theta](U)\} \\ \Downarrow \\ FT[f](X,Y) \\ \Downarrow \\ f \text{ using inverse FT} \end{array}}$$

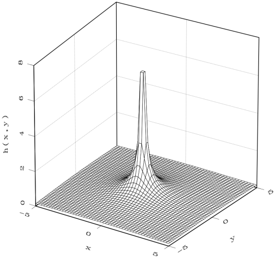
= Direct inversion (1D FT + 2D IFT)

Inversion - 2

Backprojection theorem

$$B[p](x,y) = (f * h)(x,y)$$

with $h(x,y) = \frac{1}{\sqrt{x^2+y^2}}$



$\Rightarrow$ reconstruction using deconvolution:

$$f = IFT\ [\ FT(B[p])\ .\ \rho\ ]$$

with $\rho(X,Y) = \sqrt{X^2+Y^2}$

(2D filtering and FT)

Inversion - 3

Filtered backprojection

$$f = B[\tilde{p}]$$

$$\text{with } \tilde{p}_\theta = IFT\ [\ FT[p_\theta](U)\ .\ |U|\ ]$$

$\Rightarrow$ reconstruction scheme:

filtering of projections (1D)
$\Downarrow$
backprojection of filtered projections

In practice: filtering using $H(U) = |U|\ .\ W(U)$
$W(U)$: low-pass filter
$\Rightarrow$ compromise spatial resolution / noise

Digitization

- Ideal continuous and infinite case:
  - domain $\mathbb{R}^2$
  - continuous function f
  - continuous p_θ, known $\forall\theta \in [0, \pi[$
- In practice:
  - p_θ for a finite number of θ_k (acquisition system)
  - p_{θ_k} known at discrete points u_l (detectors)
  - reconstruction of f at a finite number of points (algorithms and computation)

reconstruction:

$$\{p_{\theta_k}(u_l),\ 0 \leq l < NP, 0 \leq k < M\}$$
$$\rightarrow \quad \{f(x_i, y_j), 0 \leq i < N, 0 \leq j < N\}$$

with:

$$\theta_k = k\Delta\theta,\ \Delta\theta = \frac{\pi}{M}, \quad u_l = ld$$
$$x_i = i\Delta x, \quad y_j = j\Delta y$$

Two classes of methods in the discrete case

- Analytical methods:
    - discrete operators
    - digitization of inversion formulas

- Algebraic methods:
    - digitization of projection equation
    - solving a linear system of equations

Discrete operators

- DFT:

$$F_k = \sum_{l=0}^{N-1} f_l \exp(\frac{-2\pi}{N} lk)$$

  spectrum overlap issue $\Rightarrow$ Shannon
  $\Rightarrow$ hypothesis of limited spectrum

- Discrete backprojection:

$$B[p](x_i, y_j) = \frac{\pi}{M} \sum_{k=0}^{M-1} p_{\theta_k}(x_i \cos\theta_k + y_j \sin\theta_k)$$

$$x_i \cos\theta_k + y_j \sin\theta_k \neq u_l$$

$$\Downarrow$$

interpolation
or pre-interpolation of p_θ

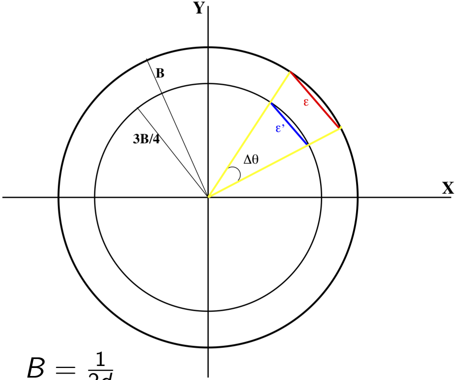
Reconstruction using direct inversion

$$DFT[p_{\theta_k}](U_l) = DFT[f](U_l \cos\theta_k, U_l \sin\theta_k)$$

$\Rightarrow$ reconstruction scheme:

$\{p_{\theta_k}(u_l)\}$
$\Downarrow$
$\{DFT[p_\theta](U_l)\}$
$\Downarrow$
estimation of $FT[f]$ in polar coordinates
$\Downarrow$
interpolation polar / Cartesian coordinates
$\Downarrow$
f using IDFT

Sampling



Projections: $d \;\Rightarrow\; B = \frac{1}{2d}$

Fourier domain:

- radial: $\rho = \frac{2B}{NP} = \frac{1}{dNP}$
- azimutal: $\varepsilon = \rho \;\Rightarrow\; \Delta\theta = \frac{2}{NP}$
- or: $\varepsilon' = \rho_{3B/4} = \frac{3}{4}B\Delta\theta = \frac{2B}{NP} \;\Rightarrow\; \Delta\theta = \frac{8}{3NP}$

  $\Rightarrow\; M(\text{number of projections})$

Reconstruction using 2D deconvolution

- discrete backprojection of all projections
- deconvolution using DFT
  - on a larger image (to avoid aliasing)
  - filter + window (to cope with noisy data)

Reconstruction using discrete filtered backprojection

Filtering of projections:

$$B = \frac{1}{2d}$$

$$\Downarrow$$

$$FT(k)(U) = \begin{cases} |U| & \text{if } |U| < B \\ 0 & \text{otherwise} \end{cases}$$

- Ramachandran and Lakshminarayanan :

$$FT(\hat{k})(U) = |U| Rect_B(U)$$

$$\Rightarrow \hat{k}(u) = 2B^2 \left(\frac{\sin(2\pi Bu)}{2\pi Bu} \right) - B^2 \left(\frac{\sin(\pi Bu)}{\pi Bu} \right)^2$$

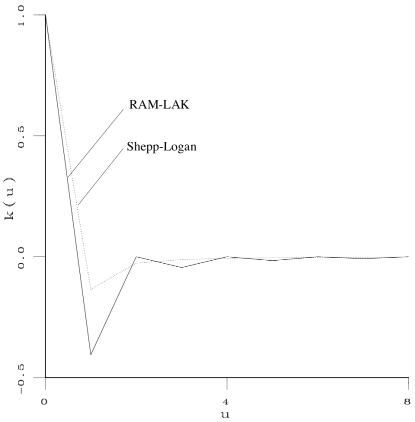
$$\Rightarrow k(\frac{m}{2B}) = \begin{cases} B^2 & \text{if} \quad m = 0 \\ 0 & \text{if} \quad m \text{ even and } \neq 0 \\ \frac{-4B^2}{m^2\pi^2} & \text{if} \quad m \text{ odd} \end{cases}$$

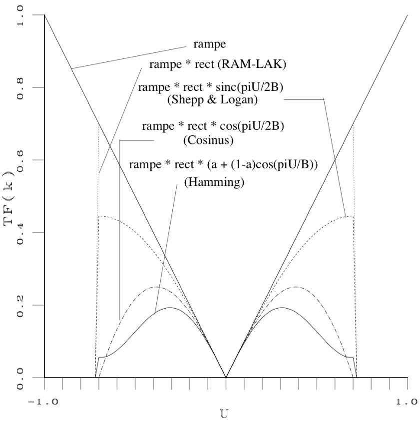
- Shepp and Logan :

$$FT(\hat{k})(U) = |U| Rect_B(U) \frac{\sin(\frac{\pi U}{2B})}{\frac{\pi U}{2B}}$$

$$\Rightarrow k(\frac{m}{2B}) = \frac{-4B^2}{\pi^2(4m^2 - 1)}$$

- Other windows: cosinus, Hamming, etc.
- Implementation:
  - discrete convolution
  - or in the Fourier domain (using FFT)
- Advantages:
  - 1D computations
  - every projection can be processed as soon as it is acquired


RAM-LAK
Shepp-Logan
k (u)
u
1.0
0.5
0.0
-0.5
0
4
8


rampe
rampe * rect (RAM-LAK)
rampe * rect * sinc(piU/2B)
(Shepp & Logan)
rampe * rect * cos(piU/2B)
(Cosinus)
rampe * rect * (a + (1-a)cos(piU/B))
(Hamming)
TF (k)
U
1.0
0.8
0.6
0.4
0.2
0.0
-1.0
1.0

Algebraic reconstruction methods

f written as:

$$f(x,y) = \sum_{i=1}^{n} f_i \varphi_i(x,y)$$

Most used basis: pixel basis

$$\varphi_i(x,y) = \begin{cases} 1 \text{ if } (x,y) = \text{pixel } i \\ 0 \text{ otherwise} \end{cases}$$

$$\Downarrow$$

$$p_j = \sum_{i=1}^{n} R_{ji} f_i$$

$$\Downarrow$$

$$p = Rf$$

with $p_j = p_{\theta_k}(u_l)$ and $R_{ji} = \int \varphi_i(u_l \cos\theta_k - v \sin\theta_k, u_l \sin\theta_k + v \cos\theta_k) dv$

- p: measurement vector (all projection values)

  size $m = M \times NP$ = number of projections $\times$ number of points / projection

- f: vectorized image values (to be computed)

  size $n = N \times N$ = number of pixels

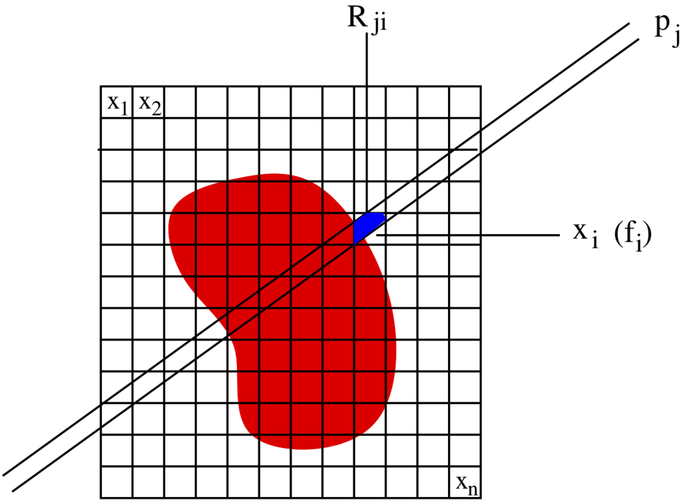
- R: projection matrix

  size $m \times n$
  depends only on the acquisition design

$$R_{ji} = \begin{cases} 1 \text{ if ray } j \text{ meets pixel } i \\ 0 \text{ otherwise} \end{cases}$$

or:

$$R_{ji} \propto \text{ overlap between ray } j \text{ and pixel } i$$


R ji
p j
x1
x2
xi (fi)
xn

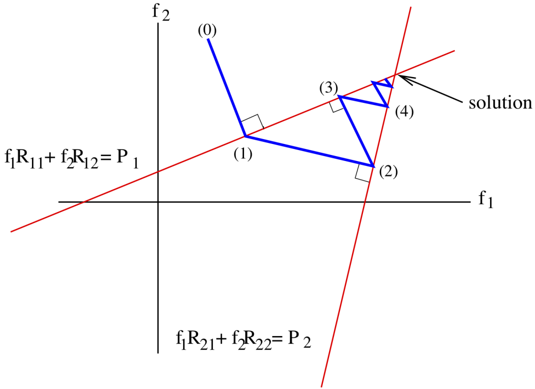
Problems with direct inversion:

- Size of the matrix (at least 250000×250000)
- A lot of 0
- Noise

$\Rightarrow$ Iterative methods

- ART: correction of f_i by using one projection at each iteration
- SIRT: correction of f_i by using all rays passing through x_i

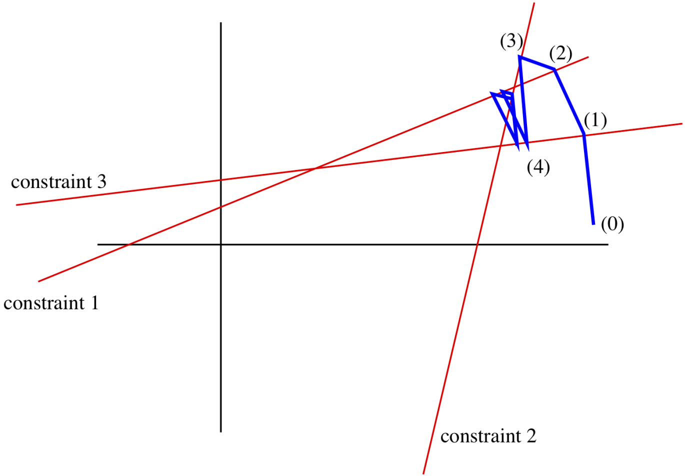
ART



$$f_i^{(k)} = f_i^{(k-1)} + R_{ji} \frac{p_j - R_j f^{(k-1)}}{\|R_j\|^2}$$

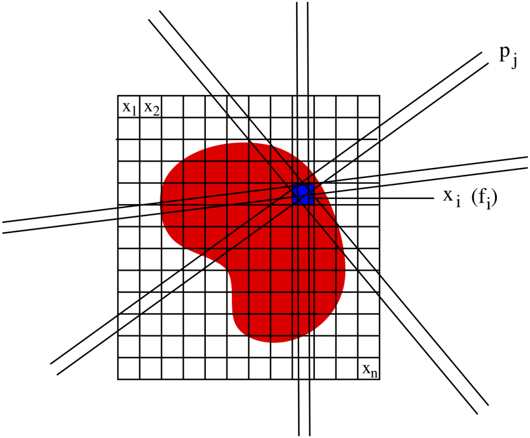
$$j = k[m] + 1$$

Noisy case



$\Rightarrow$ oscillations

SIRT



$$f_i^{(k)} = f_i^{(k-1)} + \frac{\sum_j p_j}{\sum_j \sum_i R_{ji}} - \frac{\sum_j R_j f^{(k-1)}}{\sum_j \|R_j\|^2}$$

Limitations

- Physics:
  - non-monochromatic, non infinitely thin rays
  - beam hardening
  - scattering
  - patient's movements
- Incomplete data:
  - low number of projections (e.g. cardiac imaging)
  - noisy data

$\Rightarrow$ ill-posed problem

Well-posed problem (Hadamard)

- at least one solution for each data set
- uniqueness of the solution
- the solution is a continuous function of the data

Here, for tomography: ill-posed problem

$\Rightarrow$ Regularization

Least square solution

$$Rf = p$$

but R^{-1} may not exist, may be ill-conditioned...
Approximation:

$$\min C(RF, p)$$

C: dissimilarity criterion

Least square solution:

$$f = (R^t R)^{-1} R^t p$$

if $Rank(R) = n$
otherwise infinite set of solutions
$\Rightarrow$ minimal norm solution

But can be instable / ill-conditioned

Stability analysis

σ_k^2: eigenvalues of R^tR and of RR^t ($\sigma_1 > \sigma_2 > ... \geq 0$)

$$RR^t p_k = \sigma_k^2 p_k, \quad R^t R f_k = \sigma_k^2 f_k$$

for $\sigma_k \neq 0$: $p_k = \sigma_k^{-1} R f_k, \quad f_k = \sigma_k^{-1} R^t p_k$

$$\begin{aligned} f &= (R^tR)^{-1}R^tp = (R^tR)^{-1}R^t(\sum_k < p.p_k > p_k) \\ &= (R^tR)^{-1}(\sum_k < p.p_k > \sigma_k f_k) = \sum_k < p.p_k > \sigma_k^{-1} f_k \end{aligned}$$

Noisy data $\Rightarrow$ *measures* $p + b$

$$f = \sum_k < p.p_k > \sigma_k^{-1} f_k + \sum_k < b.p_k > \sigma_k^{-1} f_k$$

High frequency noise $\Rightarrow$ large coefficients for the small eigenvalues (large values σ_k^{-1}) – cf. restoration
$\Rightarrow$ instability

Regularization

- troncate the decomposition (cf. restoration using SVD)
- weakening small eigenvalues:

$$f = \sum_k w_k \sigma_k^{-1} < p.p_k > f_k$$

- stable solution + regularity constraints

$$\min J(f) = \|Rf - p\|^2 + \gamma \Gamma(f)$$

e.g. $\Gamma(f) = \|f\|^2 \Rightarrow$

$$\boxed{f = (R^t R + \gamma I)^{-1} R^t p}$$

$$\Rightarrow f = \sum_k \frac{\sigma_k}{\sigma_k^2 + \gamma} < p.p_k > f_k$$

  - compromise precision / stability
  - introduction of other prior information in the regularization term

Extensions

Non-parallel geometry:

- Neglect divergence and use parallel approximation
  $\Rightarrow$ acceptable error if beam angle < 15 degrees
- Reorganize data into parallel projections
- Reformulate the problem:
  - projection theorem does not apply
    $\Rightarrow$ no direct reconstruction
  - adaptation of backprojection theorem
    $\Rightarrow$ similar algorithm
  - correction of filtered backprojection formulas
    $\Rightarrow$ slightly different algorithms
  - algebraic methods: adaptation of R
    $\Rightarrow$ the simplest method

Other methods:

- statistical / Bayesian approaches
- 3D
- structural approaches
- ...

A few references

- R. Gordon and G. T. Herman. Three-dimensional Reconstruction from Projections: a Review of Algorithms. International Review of Cytology, 38, 1973.
- G. T. Herman. Image Reconstruction from Projections: the Fundamentals of Computerized Tomography. Academic Press, New-York, 1980.
- A. C. Kak and M. Slaney. Principles of Computerized Tomographic Imaging. IEEE Press, New-York, 1988.
- A. K. Louis and F. Natterer. Mathematical Problems of Computerized Tomography. Proc. of the IEEE, 71(3) :390-408, 1993.
- F. Natterer. The Mathematics of Computerized Tomography. John Wiley and Sons, New-York, 1986.
- B. D. Smith. Cone Beam Tomography : Recent Advances and a Tutorial Review. Optical Engineering, 29(5) :524-534, 1990.