

Segmentation using Deformable Models

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Introduction

Deformable models = evolution of curves / surfaces to minimize an energy function for

- finding the best image partition into homogeneous regions
- finding the contours of an object

Representation:

- parametric,
- implicit, using level sets.

Criteria on

- contours
- region homogeneity
- other types of constraints:
 - regularity (internal)
 - external: balloon force, spatial relations, geometry...

Parametric active contours

- First work: Kass, Witkins et Terzopoulos (1987).
- Principle: evolution of a curve under internal and external forces (one object, parametric representation)

$$v(s) = [x(s), y(s)]^t \quad s \in [0, 1]$$

- Energy: $E_{total} = \int_0^1 (E_{int}(v(s)) + E_{image}(v(s)) + E_{ext}(v(s))) ds$
- Internal energy: regularization

$$E_{int} = \alpha(s) \left(\frac{dv}{ds} \right)^2 + \beta(s) \left(\frac{d^2v}{ds^2} \right)^2$$

control of tension (length of the curve) and of curvature (if s = curvilinear coordinate, tangent $T = \frac{dv}{ds}$, $\|T\| = 1$, and $\frac{dT}{ds} = kN$, k curvature).

- Energy from image information (gradient, information on contours):

$$E_{image} = g(\|\nabla f\|)$$

- External energy: many possibilities

- Euler-Lagrange equation:

$$\frac{\partial E}{\partial v} - \frac{d}{ds} \left(\frac{\partial E}{\partial v'} \right) + \frac{d^2}{ds^2} \left(\frac{\partial E}{\partial v''} \right) = 0$$

⇒ variational problem

$$-(\alpha v')'(s) + (\beta v'')''(s) + \nabla P(v) = 0$$

$$P(v) = E_{image}(v) + E_{ext}(v) \quad F(v) = -\nabla P(v)$$

+ limit conditions.

- Discretization using finite differences

$$V^t = [v_0^t, v_1^t, v_2^t, \dots, v_{n-1}^t]^t$$

$$\frac{\beta}{h^2} v_{i+2} - \left(\frac{\alpha}{h} + 4 \frac{\beta}{h^2} \right) v_{i+1} + \left(\frac{2\alpha}{h} + 4 \frac{6\beta}{h^2} \right) v_i - \left(\frac{\alpha}{h} + 4 \frac{\beta}{h^2} \right) v_{i-1} + \frac{\beta}{h^2} v_{i-2} = F(v_i)$$

$$AV = F$$

A: pentadiagonal matrix

Resolution (dynamic scheme with inertia)

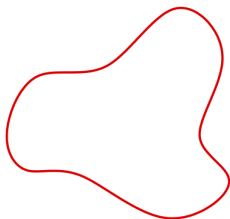
- Evolution of the curve in time:

$$\tau \frac{\partial v}{\partial t} = -\alpha v'' + \beta v'''' + F(v)$$

$$v(t+1) = (A + \tau I)^{-1}(F(v(t)) + \tau v(t))$$

- initialization (crucial for convergence)
- choice of τ (inertia, regularization of A)
- matrix inversion
- constant discretization step for t (set to have a displacement of 1-2 pixels)
- stopping criterion

Types of active contours and associated matrices

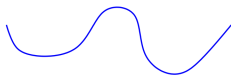


Closed active contour

$$\begin{bmatrix} 2\alpha+6\beta & -\alpha-4\beta & \beta & 0 & 0 & \beta & -\alpha-4\beta \\ -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \beta & 0 & \dots & \dots \\ \beta & -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \beta & \dots & \dots \\ 0 & \beta & -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \dots & \dots \\ 0 & 0 & \beta & -\alpha-4\beta & 2\alpha+6\beta & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

$$\begin{bmatrix} -\alpha-4\beta & \beta & 0 & 0 & \beta & -\alpha-4\beta & 2\alpha+6\beta \end{bmatrix}$$

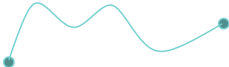
Free end-points



$$\begin{bmatrix}
 \beta & -2\beta & \beta & 0 & 0 & 0 & 0 \\
 -\alpha-2\beta & 2\alpha+5\beta & -\alpha-4\beta & \beta & 0 & \dots & \dots \\
 \beta & -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \beta & \dots & \dots \\
 0 & \beta & -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \dots & \dots \\
 0 & 0 & \beta & -\alpha-4\beta & 2\alpha+6\beta & \dots & \dots \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & 0 & \beta & -\alpha-4\beta & 2\alpha+5\beta & -\alpha-2\beta \\
 0 & 0 & 0 & \beta & -2\beta & \dots & \beta
 \end{bmatrix}$$

(with $v''(0) = v''(1) = v'''(0) = v'''(1) = 0$)

Fixed end-points



$$\begin{bmatrix}
 2\alpha+6\beta & -\alpha-4\beta & \beta & 0 & 0 & 0 & 0 \\
 -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \beta & 0 & \dots & \dots \\
 \beta & -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \beta & \dots & \dots \\
 0 & \beta & -\alpha-4\beta & 2\alpha+6\beta & -\alpha-4\beta & \dots & \dots \\
 0 & 0 & \beta & -\alpha-4\beta & 2\alpha+6\beta & \dots & \dots \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & 0 & \beta & -2\beta & \dots & \beta
 \end{bmatrix}$$

Example

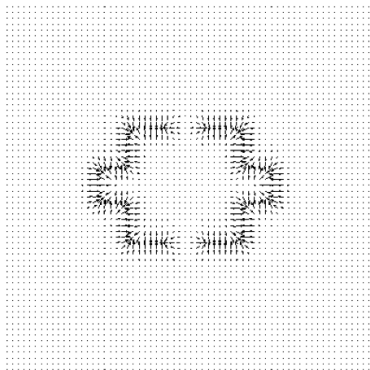
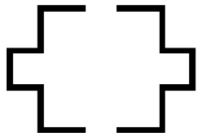


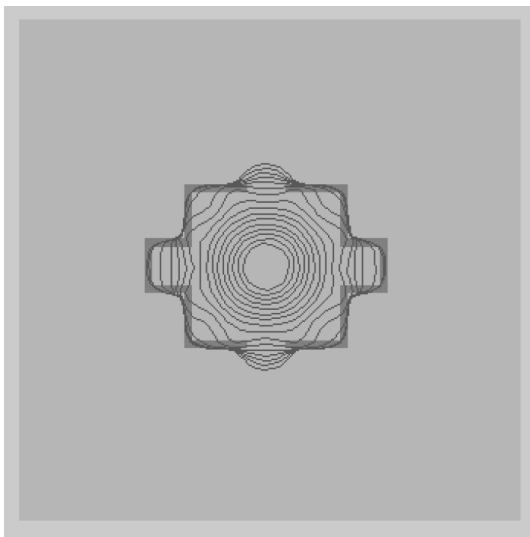
Evolution example

Evolution

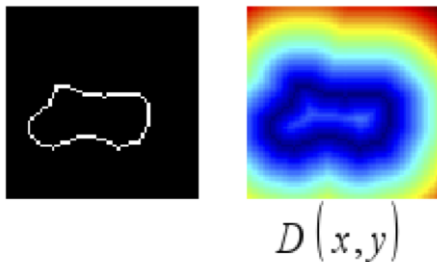
Balloon force (Cohen, 1991)

- Problem:
 - bad initialization \Rightarrow no attraction
 - no forces \Rightarrow curve collapses
- Solution using a pression force (balloon): $k_1 N(s)$
($N(s)$ unit normal vector at s).
- Initialization: inside or outside the object (not necessarily very close to the searched object)





Constraint on distance to some edges



Distance map $D(x, y) \Rightarrow$ potential

$$P_{dist}(x, y) = we^{-D(x, y)}$$

$$F_{ext} = -\nabla P_{dist}$$

Gradient vector flow (Xu et Prince, 1997)

- Objectives:
 - get rid of the constraints on the initialization
 - convergence towards concave regions
- Diffusion of gradients in the whole image
- For $\vec{v}(x, y) = (u(x, y), v(x, y))$, minimization of:

$$E = \int \int \mu(u_x^2 + u_y^2 + v_x^2 + v_y^2) + |\nabla f(x, y)|^2 |\vec{v} - \nabla f(x, y)|^2 dx dy$$

- $f =$ contour map

Solving and generalizing GVF

$$\mu \nabla^2 u - (u - f_x)(f_x^2 + f_y^2) = 0$$

$$\mu \nabla^2 v - (v - f_y)(f_x^2 + f_y^2) = 0$$

More general formulation:

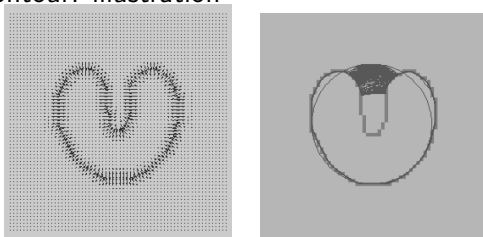
$$\frac{\partial v}{\partial t} = g(\|\nabla f\|) \nabla^2 v - h(\|\nabla f\|)(v - \nabla f)$$

$$v(x, y, 0) = \nabla f(x, y)$$

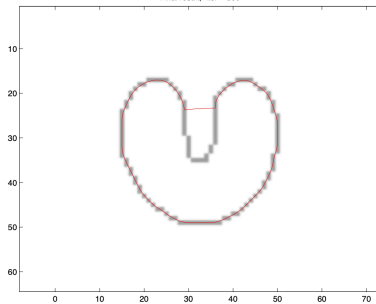
Examples for functions g and h :

- $g(r) = \mu, h(r) = r^2$
- $g(r) = \exp\left(\frac{-r^2}{k}\right), h(r) = 1 - g(r)$

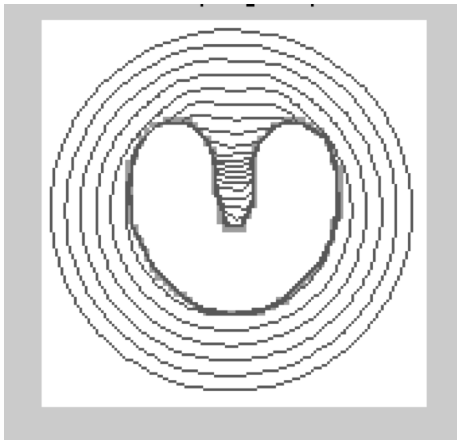
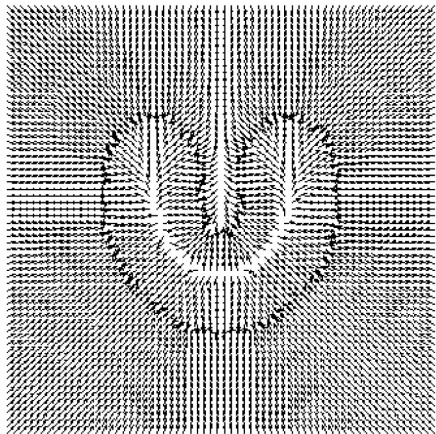
Classical active contour: illustration



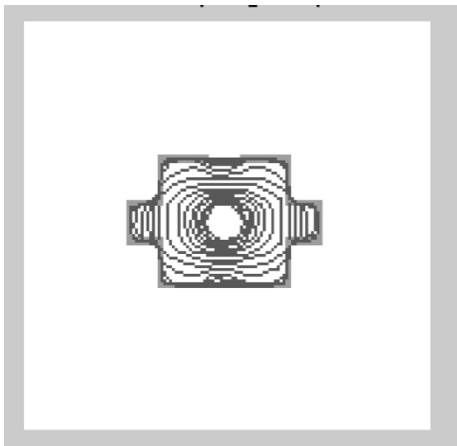
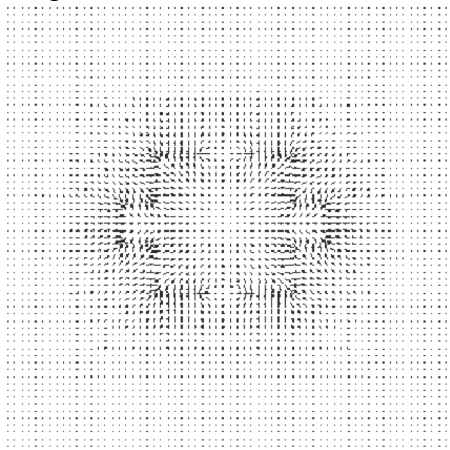
Final result, iter = 500



Using GVF:



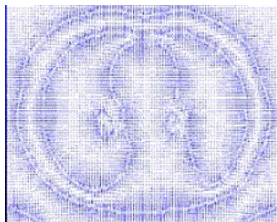
Using GVF:



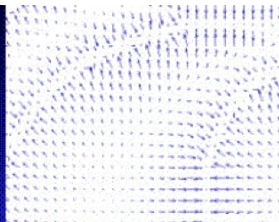
GVF: example of evolution

Evolution

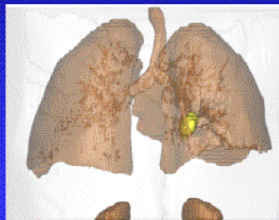
GVF: example in medical imaging

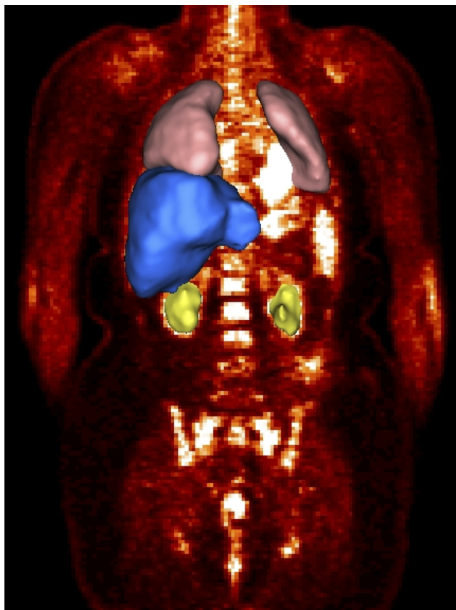
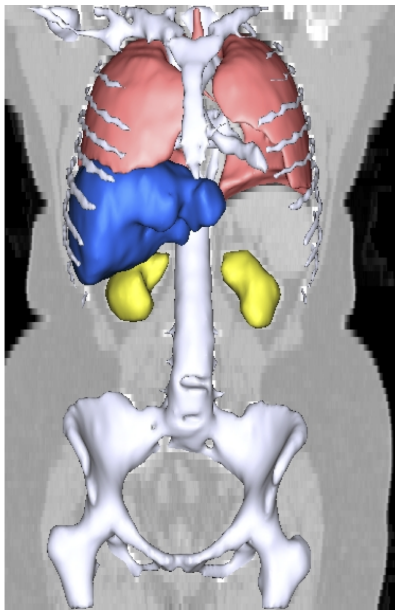


GVF field



GVF field. Detail.





3D parametric deformable models

- Segmentation with regularization:

- data fidelity
- smooth surface

- Minimization of energy:

$$\begin{aligned} E(v) = & \int_{\Omega} w_{10} \left\| \frac{\partial v}{\partial r} \right\|^2 + w_{01} \left\| \frac{\partial v}{\partial s} \right\|^2 \\ & + w_{20} \left\| \frac{\partial^2 v}{\partial r^2} \right\|^2 + w_{02} \left\| \frac{\partial^2 v}{\partial s^2} \right\|^2 + 2w_{11} \left\| \frac{\partial^2 v}{\partial r \partial s} \right\|^2 dr ds \\ & + \int_{\Omega} P(v) dr ds \end{aligned}$$

- first order: elastic membrane (curvature)
- second order: thin plate (torsion)
- P : attraction potential

Similar resolution schemes as in 2D.

Geodesic active contours (Caselles, 1997)

- Principle:

$$J_1(v) = \alpha \int_a^b |v'(s)|^2 ds + \lambda \int_a^b g(|\nabla I(v(s))|)^2 ds$$

minimization of $J_2(v) = 2\sqrt{\lambda\alpha} \int_a^b |v'(s)|g(|\nabla I(v(s))|)ds$

⇒ computation of geodesics according to a new metric (induced by the image)

- Evolution equation:

$$\frac{\partial v}{\partial t} = g(I)\kappa N - (\nabla g \cdot N)N$$

Geodesic active contours: example

Evolution

Implicit representation, without parametrization

- Principle:

Let $\Gamma(t)$ be a closed hypersurface (dimension $d - 1$)

Let ψ (dimension d) be a function taking values in \mathbb{R} with

$$\Gamma(t) = \{x \in \mathbb{R}^d \mid \psi(x, t) = 0\}$$

propagation of Γ (evolution along the normal) \Leftrightarrow propagation of ψ

- Example: distance function

- Evolution equation of ψ

$$\frac{\partial \psi}{\partial t} = -F \|\nabla \psi\|$$

NB: normal $N = \frac{\nabla \psi}{\|\nabla \psi\|}$, mean curvature $k = \operatorname{div}\left(\frac{\nabla \psi}{\|\nabla \psi\|}\right)$

Additional dimension: example of distance



- Propagation speed:

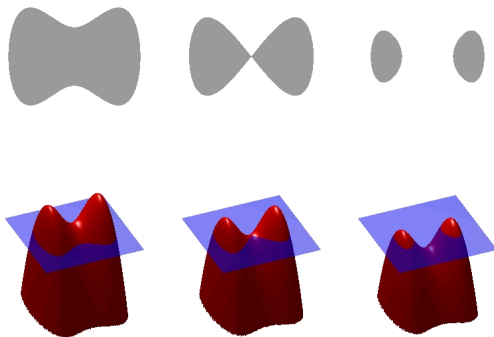
$$F = (F_A + F_G)k_I$$

- F_A expansion or contraction, independent of geometry,
- F_G geometric properties (curvature),
- k_I stopping criterion (image).

$$F = \frac{1}{1 + \|\nabla G_\sigma * I\|^p} (\pm 1 + \epsilon \kappa)$$

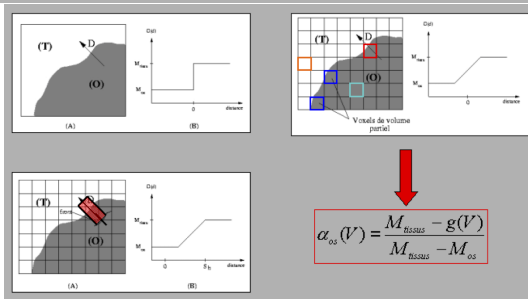
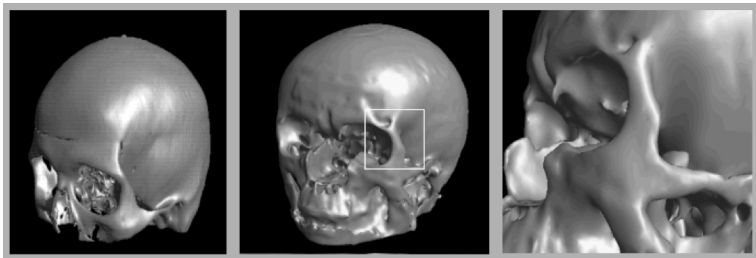
- Advantage: potential modification of topology.
- Speeding-up by computation only in a narrow band.

Level sets and change of topology

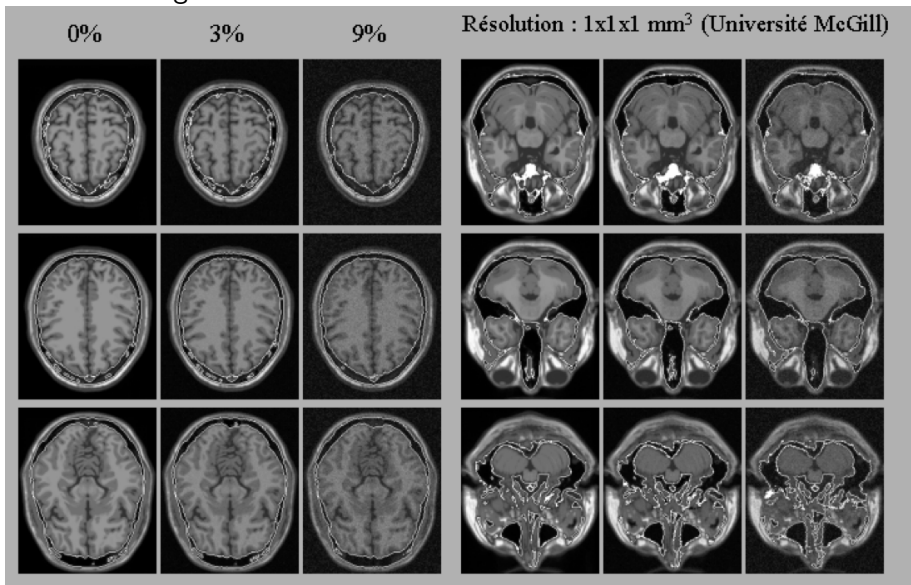


Example

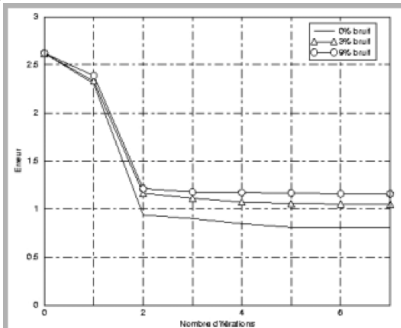
Example: bone segmentation in MRI (H. Rifai)



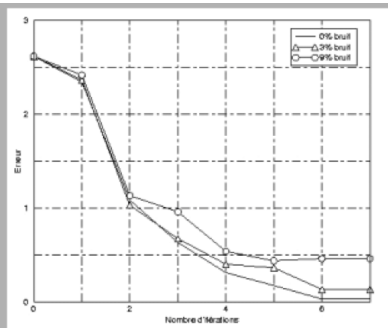
Simulated images:



Segmentation error on simulated images with various noise levels:

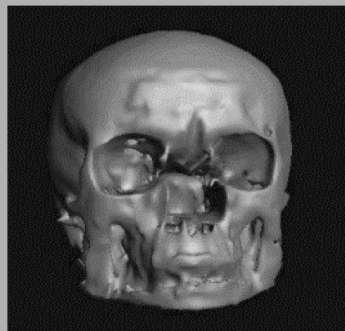
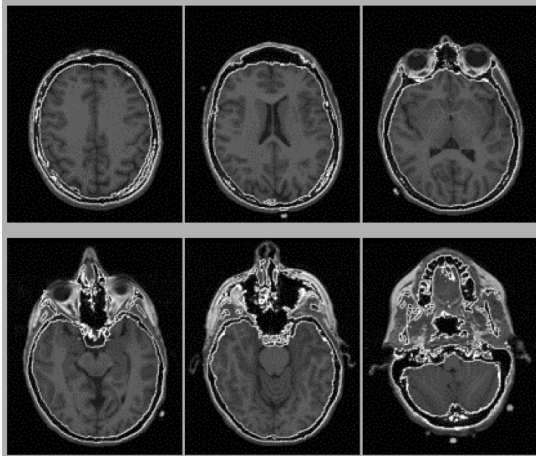


Sans estimation du volume partiel.



Avec estimation du volume partiel.

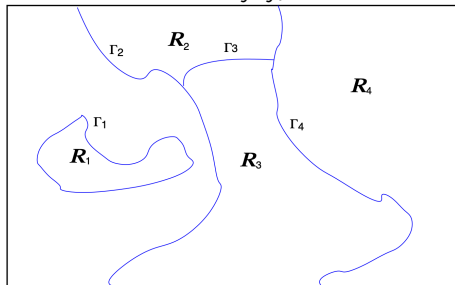
Real images:



Evolution
Result

Region-based approach: Mumford and Shah (1989)

- Image f on a support \mathcal{I} .
- Approximation by smooth functions $g(x, y)$ on regions R_i limited by contours Γ_j ($\mathcal{I} = \cup_i R_i \cup \Gamma$, $\Gamma = \cup_j \Gamma_j$):



- Functional to be minimized: $U(\Gamma, g, f) =$
$$\lambda \iint_{\mathcal{I} \setminus \Gamma} (f(x, y) - g(x, y))^2 dx dy + \mu \iint_{\mathcal{I} \setminus \Gamma} \|\nabla g(x, y)\|^2 dx dy + \nu \int_{\Gamma} dl$$

Mumford et Shah approach in the piecewise constant case

$g_i = \text{constant}$ on each R_i :

$$U_0(\Gamma, f) = \sum_i \lambda_i \iint_{R_i} (f - g_i)^2 dx dy + \nu \int_{\Gamma} dl$$

$$\Rightarrow g_i = \frac{1}{s_i} \iint_{R_i} f(x, y) dx dy$$

where s_i is the area of R_i

\Rightarrow Partition of space into homogeneous regions, characterized by their average gray level.

Formulation using level sets

- Contour Γ
- Two regions $R_1 = \text{int}(\Gamma)$ and $R_2 = \text{ext}(\Gamma)$ with constant values g_1 and g_2

$$U(\Gamma, g, f) = \lambda_1 \iint_{R_1} (f - g_1)^2 dx dy + \lambda_2 \iint_{R_2} (f - g_2)^2 dx dy + \nu \int_{\Gamma} dl$$

Level set:

$$\phi(x, y) \begin{cases} = 0 & \text{on } \Gamma \\ > 0 & \text{in } R_1 = \text{int}(\Gamma) \\ < 0 & \text{in } R_2 = \text{ext}(\Gamma) \end{cases}$$
$$\Gamma(t) = \{\phi(t) = 0\}$$

Evolution (mean curvature motion):

$$\frac{\partial \phi}{\partial t} = |\nabla \phi| \nabla \left(\frac{\nabla \phi}{|\nabla \phi|} \right) \quad \phi(0) = \phi_0$$

$$(v(r, s) = (r, s, \Phi(r, s))) \Rightarrow \frac{\partial v}{\partial r} = (1, 0, \frac{\partial \Phi}{\partial r})^t, \frac{\partial v}{\partial s} = (0, 1, \frac{\partial \Phi}{\partial s})^t, \text{ etc.}$$

$$H(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases} \quad H'(z) = \delta(z)$$

$$\Rightarrow \int_{\Gamma} dl = \int_{\mathcal{I}} |\nabla H(\phi)| dx dy = \int_{\mathcal{I}} \delta(\phi) |\nabla \phi| dx dy$$

and $U(\Gamma, g, f) =$

$$\lambda_1 \iint_{\mathcal{I}} (f - g_1)^2 H(\phi) dx dy + \lambda_2 \iint_{\mathcal{I}} (f - g_2)^2 (1 - H(\phi)) dx dy + \nu \int_{\mathcal{I}} \delta(\phi) |\nabla \phi| dx dy$$

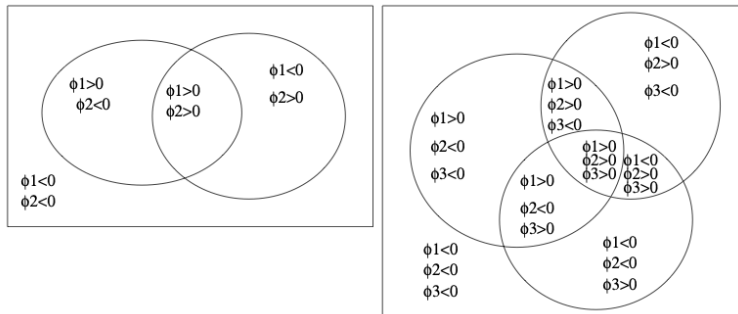
Minimization of U :

$$g_1 = \frac{\int_{\mathcal{I}} f H(\phi) dx dy}{\int_{\mathcal{I}} H(\phi) dx dy} \quad g_2 = \frac{\int_{\mathcal{I}} f (1 - H(\phi)) dx dy}{\int_{\mathcal{I}} (1 - H(\phi)) dx dy}$$

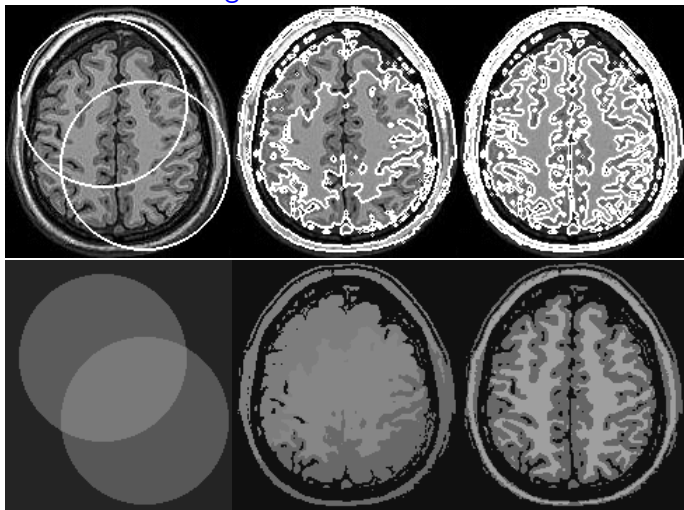
$$\frac{\partial \phi}{\partial t} = \delta(\phi) \left[\nu \nabla \left(\frac{\nabla \phi}{|\nabla \phi|} \right) - \lambda_1 (f - g_1)^2 + \lambda_2 (f - g_2)^2 \right]$$

In practice: smooth versions of δ and H .

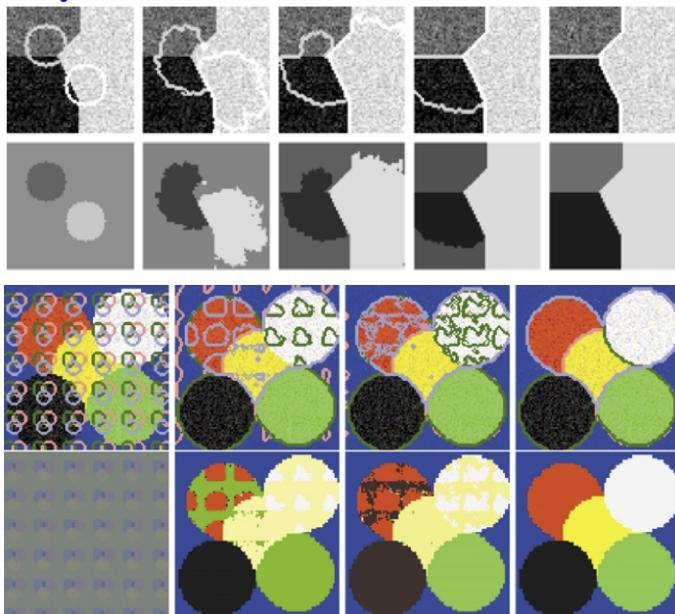
Level sets and multi-phase models (Chan, Vese, 2001, 2002)



Segmentation of a MRI image with two level sets



Example with junctions



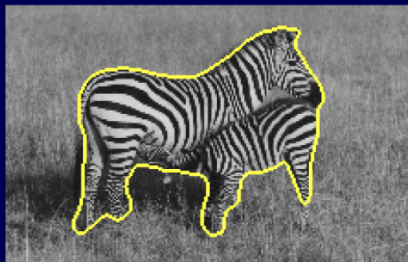
Texture segmentation

1. Generate sparse texture features
by nonlinear diffusion filtering



Brox, Weickert '04, '06

2. Mumford-Shah segmentation of vector-valued features



Brox, Weickert '04, '06

Region competition

Regions characterized by intensity distributions.

Probabilistic setting: partition $\mathcal{P}(\Omega) = \{\Omega_e, \Omega_i\}$ maximizing $\rho(I|\mathcal{P}(\Omega))\rho(\mathcal{P}(\Omega))$.

$$\rho(\mathcal{P}(\Omega)) \propto \nu \exp(-\nu|C|), \quad \nu > 0$$

$$\rho(I|\mathcal{P}(\Omega)) = \rho(I|\Omega_e)\rho(I|\Omega_i) = \prod_{x \in \Omega_e} p_e(I(x), \theta_e) \prod_{x \in \Omega_i} p_i(I(x), \theta_i)$$

$$\rho(\mathcal{P}(\Omega)|I) = \nu \exp(-\nu|C|) \prod_{x \in \Omega_e} p_e(I(x), \theta_e) \prod_{x \in \Omega_i} p_i(I(x), \theta_i)$$

Formulation as an energy minimization problem:

$$E(\{C, \theta_e, \theta_i\}) = E_{reg}(C) + E_e(\{C, \theta_e\}) + E_i(\{C, \theta_i\})$$

$$\begin{cases} E_{reg}(C) = -\log \nu + \nu|C|, \\ E_e(\{C, \theta_e\}) = -\int_{x \in \Omega_e} \log p_e(I(x), \theta_e) dx \\ E_i(\{C, \theta_i\}) = -\int_{x \in \Omega_i} \log p_i(I(x), \theta_i) dx \end{cases}$$

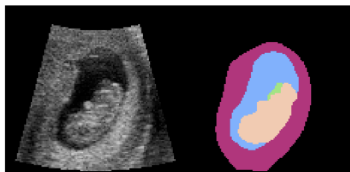
Implementation using level sets

$\phi : \Omega \rightarrow \mathbb{R}$, $\phi(x) > 0$ in Ω_e , $\phi(x) < 0$ in Ω_i and $\phi(x) = 0$ on C .

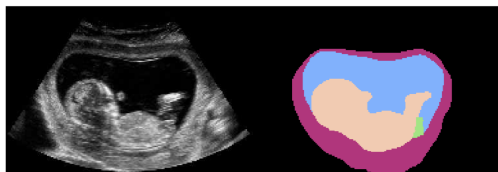
$$E(\phi, \theta_i, \theta_e) = E_{reg}(\phi) + E_e(\phi, \theta_e) + E_i(\phi, \theta_i)$$

$$\begin{cases} E_{reg}(\phi) = \nu \int_{x \in \Omega} \delta(\phi(x)) |\nabla \phi(x)| dx, \\ E_e(\phi, \theta_e) = - \int_{x \in \Omega} H(\phi(x)) \log(p_e(I(x), \theta_e)) dx \\ E_i(\phi, \theta_i) = - \int_{x \in \Omega} (1 - H(\phi(x))) \log(p_i(I(x), \theta_i)) dx \end{cases}$$

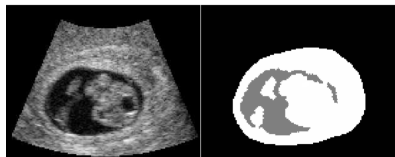
Example in ultrasound imaging (Jérémie Anquez)



(a)



(b)

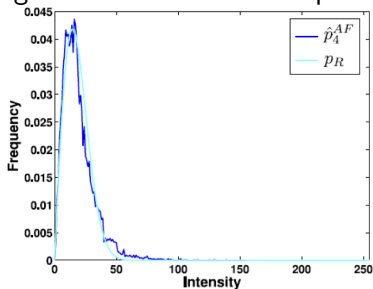


(c)

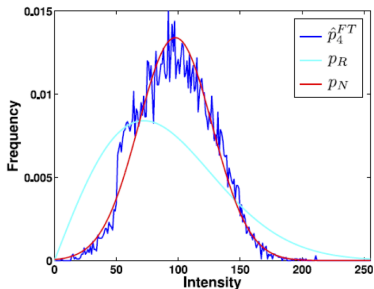


(d)

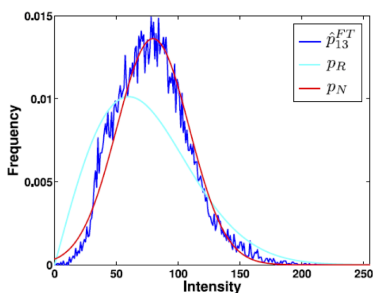
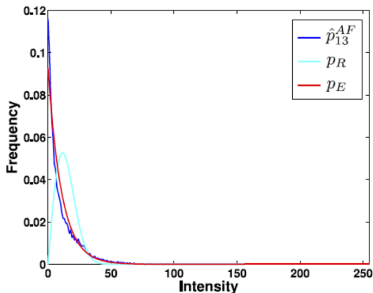
Learning distributions and their parameters:

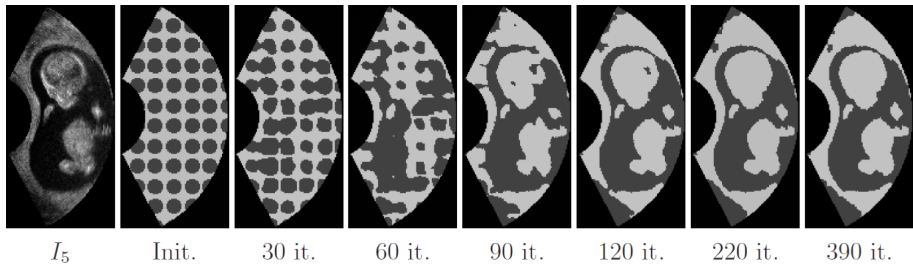


(a)



(b)



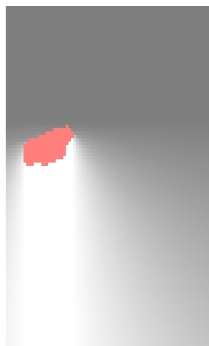


Constraining deformable models by spatial relations (Olivier Colliot et al.)

Examples

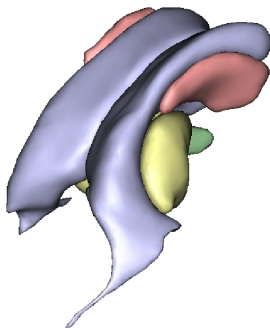
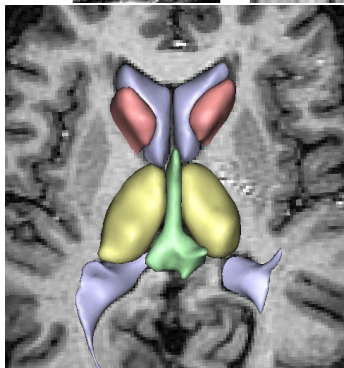


close to the lateral ventricle

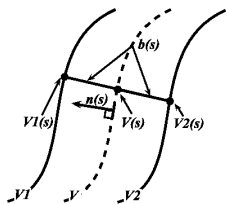


below

⇒ **additional external force** (avoids leaking in undesired regions)



Retina imaging (ISEP and XV-XX)



parallelism constraint

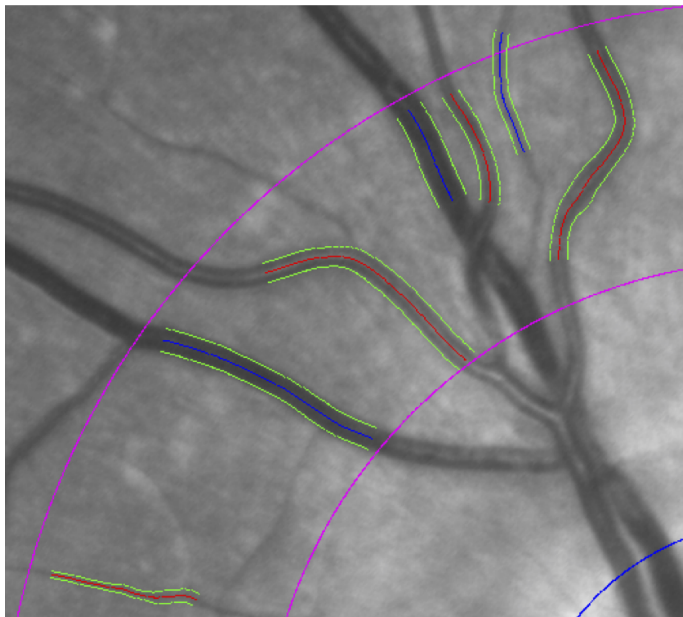
$$E(V, V_1, V_2, b) = E_{Image}(V_1) + E_{Image}(V_2) + E_{Int}(V) + R(V_1, V_2, b)$$

$$E_{Image}(V_i) = \int_0^1 P(V_i(s)) ds = - \int_0^1 |\nabla I(V_i(s))|^2 ds$$

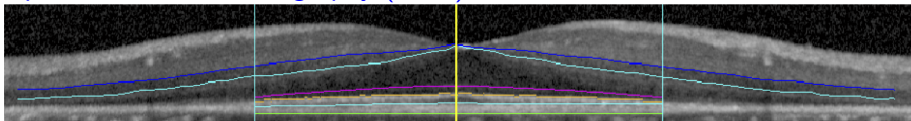
$$E_{Int}(V) = \frac{1}{2} \int \alpha(s) \left| \frac{\partial V(s, t)}{\partial s} \right|^2 + \beta(s) \left| \frac{\partial^2 V(s, t)}{\partial s^2} \right|^2 ds$$

$$R(V_1, V_2, b) = \int_0^1 \varphi(s) (b'(s))^2 ds$$

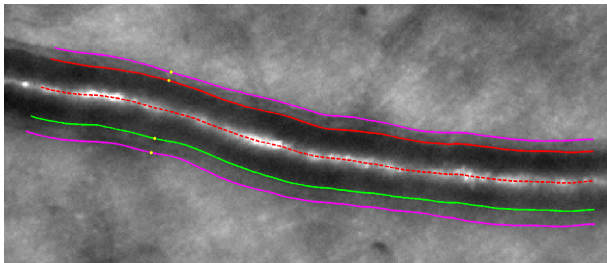
Eye fundus:



Optical coherence tomography (OCT):



Adaptive optics:



Scikit-Image - Segmentation

- `skimage.segmentation.active_contour`: parametric representation, contour-based approach.
- `skimage.segmentation.chan_vese`: implicit representation (using level sets), region-based approach.

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